036

THE OPEN UNIVERSITY OF SRI LANKA
BACHELOR OF TECHNOLOGY (level 06)
ECX 6241
FIELD THEORY
FINAL EXAMINATION 2005

DATE : 10th May 2006 **TIME : 9.30 – 12.30 hours**

Select **ONE** question each, from Sections **A** and **B** and answer **all questions** in Section **C**.

SECTION A:

Answer **ONE** question.

1. A conducting sphere of radius 'b' with a charge Q is placed in an initially uniform electric field $\mathbf{E} = \mathbf{a_z} E_0$. Determine

   (a) the potential distribution $V(r, \theta)$
   (b) the electric field intensity $\mathbf{E}(r, \theta)$ after the introduction of the sphere

   [Note: for spherical boundary value problems with no azimuthal variation
   $V_n(r, \theta) = [A_n r^n + B_n r^{-(n+1)}] P_n(\cos\theta)$, where for n=0 $P_n(\cos\theta) = 1$; for n=1
   $P_n(\cos\theta) = \cos\theta$]

2. A rectangular conducting sheet of conductivity σ , has a width 'a' and a height 'b'. The electrostatic potential at the side edges of the conducting sheet is as shown in Figure 1. Find
   (a) the potential distribution
   (b) the current density every where within the sheet.

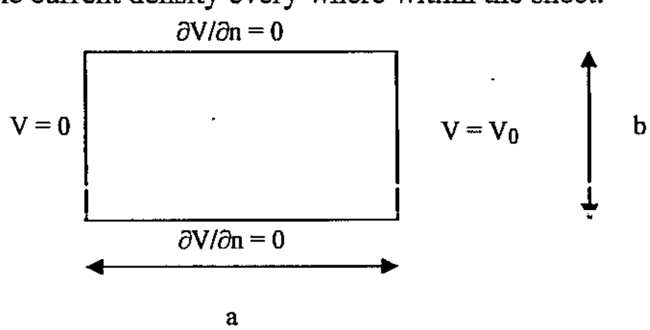


Figure 1

SECTION B:

Answer **ONE** question

3. A current of density $\mathbf{J} = \mathbf{a_z} J_0$, flows along an infinitely long solid cylindrical conductor of radius 'b' oriented along the z axis.

   (a) calculate the magnetic flux density, inside and outside the conductor
   (b) find the Poyinting vector on the surface of the conductor
   (c) verify Poynting theorem

4. The far field of an antenna could be expressed as given below in spherical coördinates:

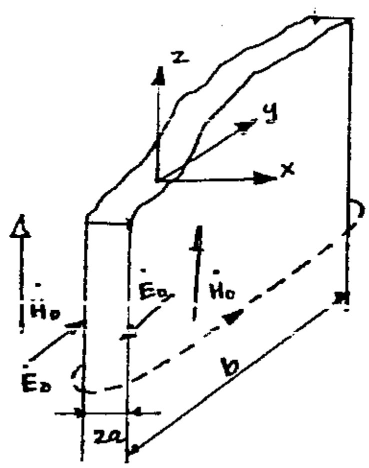
$$\mathbf{E}(r, \theta) = \mathbf{a_\theta} \, [\, (k_1/r) \sin \theta] \, e^{-j\beta r}$$
$$\mathbf{H}(r, \theta) = \mathbf{a_\varphi} \, [(k_2/r) \sin \theta] \, e^{-j\beta r}$$

   Where k_1 and k_2 are constants and other symbols carry their usual meaning.

   (a) write an expression for the instantaneous Poyinting vector
   (b) find the total average power radiated by the antenna

SECTIONC:

Answer **ALL** questions.



For the 2nd assignment in Field Theory a student considered the varying magnetic field in a thin plate. A part of his answer for this assignment is reproduced below for your reference.

A thin plate of width 'b' and thickness '2a' is considered; where b >> 2a (refer Figure 2). The conductivity of the plate is σ.

Figure2

As shown in the figure 2 the Cartesian coordinate axes are placed in such a manner that the magnetic field H_0 exterior to the plate should have only a z-component. This field is same in strength on both sides of the plate. Assuming that the plate is normal to the x-axis; we can say that the H-field has only a z-component and depends only on the x-coordinate inside the homogeneous and isotropic plate. The basic field equations which applies for this situation is

$$\nabla^2 \mathbf{H} = j\omega\mu\mu_0\sigma\mathbf{H} \text{ ....(1)}$$

For the given plate from (1) we could get

$$d^2H_z/dx^2 = \gamma^2 H_z \quad \text{at} \quad \gamma = \alpha + j\beta \text{ ....(2)}$$

The solution of equation (2) could be written as $H = A_1 e^{-\gamma x} + A_2 e^{\gamma x}$ ....(3) where H is given by two waves traveling in opposite directions ±x. By applying the boundary conditions $H_{x=a} = H_{x=-a} = H_0$

We get $A_1 = A_2 = H_0/2\cosh(\gamma a)$ ....(4)

Therefore, $H = H_0 (\cosh(\gamma x)/\cosh(\gamma a))$ ....(5)

The magnetic field distribution over the plate thickness is shown in figure 3. Where the normalized field strength $h = H/H_0$ is plotted versus the normalized coordinate x/a.

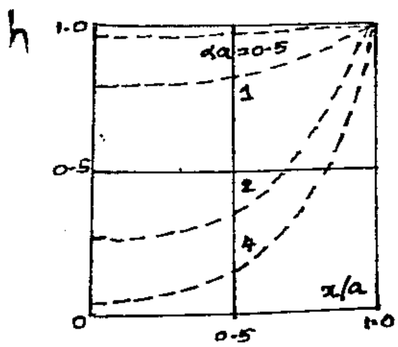


Figure 3

$$h(x) = |\ H/H_0| = |\cosh(\gamma x)/\cosh(\gamma a)|$$

using (4) ; equation (3) could be written as,

$$H = (H_0/2\cosh(\gamma a))\ e^{-\gamma x} + (H_0/2\cosh(\gamma a))\ e^{\gamma x} = H_+ + H_- \text{ ....(6)}$$

Where, H- represents the wave traveling in the negative x direction (from the right plate surface towards left) and H_+ represents the wave traveling in the positive x direction (from the left surface of the plate towards right).

The electric field **E** (or the corresponding current density $\mathbf{J} = \sigma\mathbf{E}$) could be found by using the appropriate Maxwell's equation,

$\mathbf{E} = \text{curl } \mathbf{H}/\sigma$ ....(7)

From (7) it is possible to derive for our situation,

$E = E_y = -(1/\sigma)\partial H/\partial x$ ....(8)

We could find E+ traveling in the +y direction and E- traveling in the –y direction by using $E_+ = \xi H_+$ and $E_- = \xi H_-$. Where ξ is the characteristic impedance of the medium.

By referring to the above answer following questions.

1. Starting from time-harmonic Maxwell;s equations derive equation (1), for the case of good conductors $\sigma >>> \varepsilon\varepsilon_0$

   (10marks)

2. Derive equation (2) from (1). (5 marks)

3. Starting from (3) and using appropriate boundary conditions derive (5). (5 marks)

4. (a) Referring to equation (6) and figure 3 find $|H_+ / H_-|$ for the point x = 0.5a when, $\alpha a = 4$.

   (5marks)

   (b) Is it possible to neglect one wave H_+ or H- (one compared to the other) under these conditions? Explain. (5marks)

5. (a) Find the distance traveled by H_+ and H- separately in the situation considered in question 4. (10marks)

   (b) Is it possible to say that only one wave (H_+ or H-) enters the region $x \geq 0.5a$? Explain. (5marks)

6. Show the derivation of equation (8). (5marks)

7. Write expressions for corresponding Poyinting vectors $\mathbf{P}_+$ and $\mathbf{P}_-$. What can you say about the direction of each Poyinting vector? (10marks)

VECTOR RELATIONS

UNIT VECTORS

$\mathbf{a}_x, \mathbf{a}_y, \mathbf{a}_z$ — rectangular
$\mathbf{a}_\rho, \mathbf{a}_\phi, \mathbf{a}_z$ — cylindrical
$\mathbf{a}_r, \mathbf{a}_\theta, \mathbf{a}_\phi$ — spherical

COORDINATE TRANSFORMATIONS

$$x = \rho \cos\phi = r \sin\theta \cos\phi$$
$$y = \rho \sin\phi = r \sin\theta \sin\phi$$
$$z = r \cos\theta$$
$$\rho = \sqrt{x^2 + y^2} = r \sin\theta$$
$$\phi = \tan^{-1} y/x$$
$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{\rho^2 + z^2}$$
$$\theta = \tan^{-1}(\sqrt{x^2 + y^2}/z) = \tan^{-1}(\rho/z)$$

COORDINATE COMPONENT TRANSFORMATIONS

$$A_x = A_\rho \cos\phi - A_\phi \sin\phi$$
$$= A_r \sin\theta \cos\phi + A_\theta \cos\theta \cos\phi - A_\phi \sin\phi$$
$$A_y = A_\rho \sin\phi + A_\phi \cos\phi$$
$$= A_r \sin\theta \sin\phi + A_\theta \cos\theta \sin\phi + A_\phi \cos\phi$$
$$A_z = A_r \cos\theta - A_\theta \sin\theta$$
$$A_\rho = A_x \cos\phi + A_y \sin\phi = A_r \sin\theta + A_\theta \cos\theta$$
$$A_\phi = -A_x \sin\phi + A_y \cos\phi$$
$$A_r = A_x \sin\theta \cos\phi + A_y \sin\theta \sin\phi + A_z \cos\theta$$
$$= A_\rho \sin\theta + A_z \cos\theta$$
$$A_\theta = A_x \cos\theta \cos\phi + A_y \cos\theta \sin\phi - A_z \sin\theta$$
$$= A_\rho \cos\theta - A_z \sin\theta$$

DIFFERENTIAL ELEMENTS OF VECTOR LENGTH

$$d\mathbf{l} = \begin{cases} \mathbf{a}_x\, dx + \mathbf{a}_y\, dy + \mathbf{a}_z\, dz \\ \mathbf{a}_\rho\, d\rho + \mathbf{a}_\phi \rho\, d\phi + \mathbf{a}_z\, dz \\ \mathbf{a}_r\, dr + \mathbf{a}_\theta r\, d\theta + \mathbf{a}_\phi r \sin\theta\, d\phi \end{cases}$$

DIFFERENTIAL ELEMENTS OF VECTOR AREA

$$d\mathbf{s} = \begin{cases} \mathbf{a}_x\, dy\, dz + \mathbf{a}_y\, dx\, dz + \mathbf{a}_z\, dx\, dy \\ \mathbf{a}_\rho \rho\, d\phi\, dz + \mathbf{a}_\phi\, d\rho\, dz + \mathbf{a}_z \rho\, d\rho\, d\phi \\ \mathbf{a}_r r^2 \sin\theta\, d\theta\, d\phi + \mathbf{a}_\theta r \sin\theta\, dr\, d\phi + \mathbf{a}_\phi r\, dr\, d\theta \end{cases}$$

DIFFERENTIAL ELEMENTS OF VOLUME

$$dv = \begin{cases} dx\, dy\, dz \\ \rho\, d\rho\, d\phi\, dz \\ r^2 \sin\theta\, dr\, d\theta\, d\phi \end{cases}$$

VECTOR OPERATIONS—RECTANGULAR COORDINATES

$$\nabla\alpha = \mathbf{a}_x \frac{\partial \alpha}{\partial x} + \mathbf{a}_y \frac{\partial \alpha}{\partial y} + \mathbf{a}_z \frac{\partial \alpha}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \mathbf{a}_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \mathbf{a}_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \mathbf{a}_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla^2\alpha = \frac{\partial^2 \alpha}{\partial x^2} + \frac{\partial^2 \alpha}{\partial y^2} + \frac{\partial^2 \alpha}{\partial z^2} = \nabla \cdot \nabla\alpha$$

$$\nabla^2\mathbf{A} = \mathbf{a}_x \nabla^2 A_x + \mathbf{a}_y \nabla^2 A_y + \mathbf{a}_z \nabla^2 A_z = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A})$$

VECTOR OPERATIONS—CYLINDRICAL COORDINATES

$$\nabla\alpha = \mathbf{a}_\rho \frac{\partial \alpha}{\partial \rho} + \mathbf{a}_\phi \frac{1}{\rho}\frac{\partial \alpha}{\partial \phi} + \mathbf{a}_z \frac{\partial \alpha}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \mathbf{a}_\rho\left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \mathbf{a}_\phi\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right) + \mathbf{a}_z \frac{1}{\rho}\left(\frac{\partial}{\partial \rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi}\right)$$

$$\nabla^2\alpha = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \alpha}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \alpha}{\partial \phi^2} + \frac{\partial^2 \alpha}{\partial z^2}$$

$$\nabla^2\mathbf{A} = \mathbf{a}_\rho\left(\nabla^2 A_\rho - \frac{2}{\rho^2}\frac{\partial A_\phi}{\partial \phi} - \frac{A_\rho}{\rho^2}\right) + \mathbf{a}_\phi\left(\nabla^2 A_\phi + \frac{2}{\rho^2}\frac{\partial A_\rho}{\partial \phi} - \frac{A_\phi}{\rho^2}\right) + \mathbf{a}_z \nabla^2 A_z$$

Vector Operations – Spherical coordinates

$$\nabla \alpha = a_r \cdot \frac{\partial \alpha}{\partial r} + a_\theta \frac{1}{r} \frac{\partial \alpha}{\partial \theta} + a_\phi \cdot \frac{1}{r \sin\theta} \cdot \frac{\partial \alpha}{\partial \phi}$$

$$\nabla \cdot A = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta}\left(A_\theta \cdot \sin\theta\right) + \frac{1}{r \sin\theta} \frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times A = \frac{1}{r^2 \sin\theta} \begin{vmatrix} a_r & r \cdot a_\theta & r \sin\theta \cdot a_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ A_r & r \cdot A_\theta & r \sin\theta \cdot A_\phi \end{vmatrix}$$

$$\nabla^2 \alpha = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \alpha}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \cdot \frac{\partial \alpha}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \alpha}{\partial \phi^2}$$

$$\nabla \times \nabla \times A = \nabla(\nabla \cdot A) - \nabla^2 A$$

$$\frac{d}{dx}(\sin ax) = a \cos x$$

$$\frac{d}{dx}(\cos ax) = -a \sin x$$

$$\int \sin ax \cdot dx = -(\cos ax)/a$$

$$\int \cos ax \cdot dx = \sin ax / a$$

$\int_V \nabla \cdot \bar{A}\, dV = \oint_S \bar{A} \cdot ds$ $\qquad \int_S (\nabla \times \bar{A}) \cdot ds = \oint_C \bar{A} \cdot dl$

$\bar{D} = \varepsilon \bar{E}, \; \bar{B} = \mu \bar{H}, \; \bar{J} = \sigma \bar{E}$

$\nabla \times (\nabla V) \equiv 0 \; ; \; \nabla \cdot (\nabla \times \bar{A}) \equiv 0 \; ; \; \bar{F} = -\nabla V + \nabla \times \bar{A}$

$\nabla \times \bar{E} = -\partial \bar{B}/\partial t \qquad \oint_C \bar{E} \cdot dl = -d\phi/dt \qquad E_{1t} = E_{2t}$

$\nabla \times \bar{H} = \bar{J} + \partial \bar{D}/\partial t \qquad \oint_C \bar{H} \cdot dl = I + \int_S \partial \bar{D}/\partial t \cdot ds \qquad B_{1n} = B_{2n}$

$\nabla \cdot \bar{D} = \rho \qquad \oint_S \bar{D} \cdot ds = Q \qquad \bar{a}_{n2} \times (\bar{H}_1 - \bar{H}_2) = \bar{J}_s$

$\nabla \cdot \bar{B} = 0 \qquad \oint_S \bar{B} \cdot ds = 0 \qquad \bar{a}_{n2} \cdot (\bar{D}_1 - \bar{D}_2) = \rho_s$

$\nabla \cdot \bar{J} + \partial \rho/\partial t = 0$

$\bar{B} = \nabla \times \bar{A} \qquad \nabla \cdot \bar{A} + \mu\varepsilon\, \partial V/\partial t = 0 \qquad \nabla^2 \bar{A} - \mu\varepsilon \frac{\partial^2 \bar{A}}{\partial t^2} = -\mu \bar{J}$

$\bar{E} = -\nabla V - \frac{\partial \bar{A}}{\partial t} \qquad \nabla^2 V - \mu\varepsilon\, \partial^2 V/\partial t^2 = -\rho/\varepsilon$

$V(R,t) = \frac{1}{4\pi\varepsilon} \int_{V'} \frac{\rho(t - R/u)}{R} dv' \qquad V(R) = \frac{1}{4\pi\varepsilon} \int_{V'} \frac{\rho e^{-jkR}}{R} \cdot dv'$

$\bar{A}(R,t) = \mu/4\pi \int_{V'} \frac{\bar{J}(t - R/u)}{R} dv' \qquad \bar{A}(R) = \mu/4\pi \int_{V'} \frac{\bar{J} \cdot e^{-jkR}}{R} dv'$

$\nabla \times \bar{E} = -j\omega\mu \bar{H} \qquad \nabla^2 V + k^2 V = -\rho/\varepsilon$

$\nabla \times \bar{H} = \bar{J} + j\omega\varepsilon \bar{E} \qquad \nabla^2 \bar{A} + k^2 \bar{A} = -\mu \bar{J} \qquad \beta = k = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{u}$

$\nabla \cdot \bar{E} = \rho/\varepsilon$

$\nabla \cdot \bar{H} = 0 \qquad \nabla \cdot \bar{A} + j\omega\mu\varepsilon V = 0 \qquad c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \qquad \eta = \sqrt{\frac{\mu}{\varepsilon}}$

$\gamma = \alpha + j\beta \; ; \; \delta = 1/\alpha \qquad \alpha = \sqrt{\pi f \mu \sigma} \qquad k = 2\pi/\lambda$

$u = f\lambda = \omega/\beta$

$\frac{\sin\theta_t}{\sin\theta_i} = n_1/n_2 \; ; \; n = c/u_p \qquad Z_0 = \left(\frac{R + j\omega L}{G + j\omega C}\right)^{1/2} \qquad \Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$

$\bar{P} = \bar{E} \times \bar{H} \qquad \gamma = [(R + j\omega L)(G + j\omega C)]^{1/2}$

$P_{av.} = \frac{1}{2} \mathrm{Re}(\bar{E} \times \bar{H}^*) \qquad \tau = 1 + \Gamma$

total radiated power $= \frac{P_{av.} \times 4\pi r^2}{D} \qquad H = \frac{1}{\eta} E \qquad S = \frac{1 + |\Gamma|}{1 - |\Gamma|}$

$R_r = 80\pi^2 \left(\frac{dl}{\lambda}\right)^2 \qquad \frac{\sin\theta_t}{\sin\theta_i} = \sqrt{\frac{\varepsilon_{r1}}{\varepsilon_{r2}}} \qquad \mu_0 = 4\pi \times 10^{-7}\ H/m$

$\Gamma_\perp = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} \; ; \; \Gamma_\parallel = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i} \qquad \varepsilon_0 = 8.854 \times 10^{-12}\ F/m$

$\tau_\perp = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} \; ; \; \tau_\parallel = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$

$\sin\theta_{B\perp} = 1/(1 + \mu_1/\mu_2)^{1/2} \; ; \; \sin\theta_{B\parallel} = 1/(1 + \varepsilon_1/\varepsilon_2)^{1/2}$

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