THE OPEN UNIVERSITY OF SRI LANKA
BACHELOR OF TECHNOLOGY (level 05)
ECX 5241
DISTRIBUTED PARAMETER SYSTEMS
FINAL EXAMINATION 2005

DATE : 19th April 2006 **TIME : 13.30 – 16.30 hours**

Select **ONE** question each from Sections **B** and **C** and answer **all questions** in Section A.

SECTION A:

Answer **ALL** questions

Read the following note and answer questions given at the end.

Analogy between diffusion, heat flow and good electrical conductors ....

Diffusion processes are dealt with in a number of branches in physics. In general they can be explained as follows: if the homogeneity of a certain medium is disturbed due to the accumulation of particles with higher volume density n, then these particles move in the direction of particles with lower density, tending to spread over uniformly in the space. The velocity of the particles **v** is proportional to the density gradient, negative in sign.

The velocity **v** is equal to the number of particles transferred in a unit time, through a unit area of the surface normal to the vector **v.** If the particles under consideration, do not vanish, nor originate, their density changes only as a consequence of their motion. Then the divergence of velocity **v** is equal to the time rate of decrease of the particle density. Now it is possible to derive expressions which give a connection between space and time rates of change for both particle density and particle flow vector separately.

We can get similar equations, if we consider the processes of temperature distribution, namely the process of heat transfer due to thermal conduction. The vector of thermal flux density **Q** is proportional to the temperature gradient and is in the opposite direction. The thermal energy density at any point is expressed as the product of temperature T and thermal capacity c. If there is no heat generation in a medium (i.e. no increase in the thermal energy density due to chemical processes or heating of the substance by electric current) then it is only the heat flux density **Q** that changes the thermal energy density. Expressing the density of thermal (free) energy as a product of thermal capacity and temperature we obtain the formula:

$$c\, \partial T/\partial t = -\mathrm{div}\, \mathbf{Q}$$

which says that the time rate of rise in thermal energy density, $c\, \partial T/\partial t$ is equal to the rate of heat inflow $-\mathrm{div}\mathbf{Q}$. When the divergence is positive, the temperature drops since the outflow of heat, $\mathrm{div}\mathbf{Q} > 0$ occurs as a result of the 'decrease' in thermal energy density.

Time varying electromagnetic fields in free space could be described by following Maxwell's equations:

$$\mathrm{curl}\, \mathbf{H} = \mathbf{J} + \partial \mathbf{D}/\partial t \;\ldots\ldots\ldots.(1)$$
$$\mathrm{curl}\, \mathbf{E} = -\partial \mathbf{B}/\partial t$$
$$\mathrm{div}\, \mathbf{B} = 0$$
$$\mathrm{div}\, \mathbf{E} = 0$$

Since $\mathbf{J} = \sigma\, \mathbf{E}$, $\mathbf{D} = \varepsilon_0 \varepsilon \mathbf{E}$ and $\mathbf{B} = \mu_0 \mu \mathbf{H}$ we could write equation (1) as follows:

$$\mathrm{curl}\, H = \sigma\, \mathbf{E} + \varepsilon_0\, \varepsilon\, \partial \mathbf{E}/\partial t$$
$$= (\sigma + \varepsilon_0\, \varepsilon\, \partial\ /\partial t)\, \mathbf{E} \;\ldots\ldots\ldots.(2)$$

In the case of good conductors, the conductivity $\sigma \gg \varepsilon_0 \varepsilon$ and we could write equation (2) as $\mathrm{curl}\, \mathbf{H} = \sigma\, \mathbf{E}\ldots\ldots\ldots(3)$

Using equation (3) and the remaining of Maxwell's equations it is possible to derive expressions for both **E** and **H**, which gives a relationship between space rates and time rates of change as earlier done in the case of diffusion and heat conduction.

Answer **ALL** questions.

1. Write an equation which gives a relationship between **v** and n. (Refer the underlined sentence in paragraph 1)

   (3 marks)

2 Write an equation which relates div **v** and $\partial n/\partial t$. (Refer the underlined sentence in paragraph 2)

   (2 marks)

3. From above written equations in questions 1 and 2 derive the differential diffusion equation for n.

   (10 marks)

4. From the equations written for questions 1 and 2 derive an equation which gives a connection between time and space rates of change for the velocity vector **v**.

(5 marks)

5. If the thermal conductivity is given by k and temperature T, express **Q** in terms of k and T.

(5 marks)

6. By considering the processes of heat transfer due to thermal conductivity derive a relationship between time and space rates of change for
   (a) T (5 marks)
   (b) **Q** (5 marks)

7. Derive expressions which relates the time and space rates of change, in the case of good conductors for

   (a) the electric field **E** (10 marks)
   (b) the magnetic field **H** (10 marks)

8. Is there a similarity between equations derived in questions 3, 4, 6 and 7, in the context of what you have learned in distributed parameter systems? Name the type of the equation that you get in all these situations.

(5 marks)

SECTION B:

Answer **ONE** question

1.

(a) The electrostatic potential at a point r, which is situated at a large distance from two point charges; -q at the origin and +q at z = a; (where r >>a) is given by

$$V = kqa \cos\theta / r^2$$

Here θ is the angle between the z-axis and the line which join the points (0,0,a/2) and P. If the electric field intensity **E** is given by **E** = -grad V, find **E**.

(b) Gauss's law of electromagnetism says $\int E.ds = q_{enclosed} / \varepsilon_0$. If the electric field is given by $\mathbf{E} = A\,\mathbf{e_r}$ where A is a constant, what is the surface integral of E over the whole closed surface of a sphere, if it has a radius 'a' and the origin is situated at (0,0,0)? What is the charge enclosed by this sphere?

(20 marks)

2. Verify whether the divergence theorem holds for the field $\mathbf{F} = 2\,\rho^2\,\mathbf{a_\rho} + z\,\mathbf{a_z}$ when the closed surface is a cylinder $\rho = 4$, $0 \le z \le 3$

(20 marks)

SECTION C:

. Answer **ONE** question

1. For a certain spherical charge distribution of radius ‘a’, the electric field intensity and the scalar potential is given by following expressions:

   for $r \leq a$ $\quad \mathbf{E}_i = \mathbf{a}_r\, \rho r/3\, \varepsilon_0 \quad$ and $\quad \Phi_i = -\rho r^2/6\, \varepsilon_0 + k_1$

   for $r \geq a$ $\quad \mathbf{E}_e = \mathbf{a}_r\, Q_0/4\pi\, \varepsilon_0\, r^2 \quad$ and $\quad \Phi_e = Q_0 / 4\pi\, \varepsilon_0\, r + k_2$

   where $Q_0 = 4\pi\, a^3 \rho\, /3$; k_1 and k_2 are constants and other symbols carry their usual meaning. Also in the electrostatic case $\mathbf{E} = -\text{grad}\ \Phi$.

   By finding the divergence of the electric field and the Laplacian of the potential or otherwise show that above expressions correspond to a situation with a constant charge density ρ inside the sphere and to the absence of charge outside the sphere.

   (20 marks)

2. Inside a long magnetized core of radius r_0, there exist a magnetic field parallel to the core axis $\mathbf{B} = B_z\, \mathbf{e}_z$. Outside the core i.e. at $r > r_0$, the magnetic field is not present.

   (a) By applying Stoke’s theorem, find the magnetic vector potential **A**, inside and outside of the core. [Note: **B** = curl **A**]

   (b) Verify by direct different ion that curl **A** is equal to the specified value of **B**.

   (20 marks)

VECTOR RELATIONS

UNIT VECTORS

$\mathbf{a}_x, \mathbf{a}_y, \mathbf{a}_z$ — rectangular
$\mathbf{a}_\rho, \mathbf{a}_\phi, \mathbf{a}_z$ — cylindrical
$\mathbf{a}_r, \mathbf{a}_\theta, \mathbf{a}_\phi$ — spherical

COORDINATE TRANSFORMATIONS

$$x = \rho\cos\phi = r\sin\theta\cos\phi$$
$$y = \rho\sin\phi = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$
$$\rho = \sqrt{x^2+y^2} = r\sin\theta$$
$$\phi = \tan^{-1} y/x$$
$$r = \sqrt{x^2+y^2+z^2} = \sqrt{\rho^2+z^2}$$
$$\theta = \tan^{-1}(\sqrt{x^2+y^2}/z) = \tan^{-1}(\rho/z)$$

COORDINATE COMPONENT TRANSFORMATIONS

$$A_x = A_\rho\cos\phi - A_\phi\sin\phi = A_r\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$$
$$A_y = A_\rho\sin\phi + A_\phi\cos\phi = A_r\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$$
$$A_z = A_r\cos\theta - A_\theta\sin\theta$$
$$A_\rho = A_x\cos\phi + A_y\sin\phi = A_r\sin\theta + A_\theta\cos\theta$$
$$A_\phi = -A_x\sin\phi + A_y\cos\phi$$
$$A_r = A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta = A_\rho\sin\theta + A_z\cos\theta$$
$$A_\theta = A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta = A_\rho\cos\theta - A_z\sin\theta$$

DIFFERENTIAL ELEMENTS OF VECTOR LENGTH

$$d\mathbf{l} = \begin{cases} \mathbf{a}_x\,dx + \mathbf{a}_y\,dy + \mathbf{a}_z\,dz \\ \mathbf{a}_\rho\,d\rho + \mathbf{a}_\phi\rho\,d\phi + \mathbf{a}_z\,dz \\ \mathbf{a}_r\,dr + \mathbf{a}_\theta r\,d\theta + \mathbf{a}_\phi r\sin\theta\,d\phi \end{cases}$$

DIFFERENTIAL ELEMENTS OF VECTOR AREA

$$d\mathbf{s} = \begin{cases} \mathbf{a}_x\,dy\,dz + \mathbf{a}_y\,dx\,dz + \mathbf{a}_z\,dx\,dy \\ \mathbf{a}_\rho\rho\,d\phi\,dz + \mathbf{a}_\phi\,d\rho\,dz + \mathbf{a}_z\rho\,d\rho\,d\phi \\ \mathbf{a}_r r^2\sin\theta\,d\theta\,d\phi + \mathbf{a}_\theta r\sin\theta\,dr\,d\phi + \mathbf{a}_\phi r\,dr\,d\theta \end{cases}$$

DIFFERENTIAL ELEMENTS OF VOLUME

$$dv = \begin{cases} dx\,dy\,dz \\ \rho\,d\rho\,d\phi\,dz \\ r^2\sin\theta\,dr\,d\theta\,d\phi \end{cases}$$

VECTOR OPERATIONS—RECTANGULAR COORDINATES

$$\nabla\alpha = \mathbf{a}_x\frac{\partial\alpha}{\partial x} + \mathbf{a}_y\frac{\partial\alpha}{\partial y} + \mathbf{a}_z\frac{\partial\alpha}{\partial z}$$
$$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$
$$\nabla\times\mathbf{A} = \mathbf{a}_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \mathbf{a}_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \mathbf{a}_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$
$$\nabla^2\alpha = \frac{\partial^2\alpha}{\partial x^2} + \frac{\partial^2\alpha}{\partial y^2} + \frac{\partial^2\alpha}{\partial z^2} = \nabla\cdot\nabla\alpha$$
$$\nabla^2\mathbf{A} = \mathbf{a}_x\nabla^2 A_x + \mathbf{a}_y\nabla^2 A_y + \mathbf{a}_z\nabla^2 A_z = \nabla(\nabla\cdot\mathbf{A}) - \nabla\times(\nabla\times\mathbf{A})$$

VECTOR OPERATIONS—CYLINDRICAL COORDINATES

$$\nabla\alpha = \mathbf{a}_\rho\frac{\partial\alpha}{\partial\rho} + \mathbf{a}_\phi\frac{1}{\rho}\frac{\partial\alpha}{\partial\phi} + \mathbf{a}_z\frac{\partial\alpha}{\partial z}$$
$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$
$$\nabla\times\mathbf{A} = \mathbf{a}_\rho\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right) + \mathbf{a}_\phi\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right) + \mathbf{a}_z\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial\phi}\right)$$
$$\nabla^2\alpha = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\alpha}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\alpha}{\partial\phi^2} + \frac{\partial^2\alpha}{\partial z^2}$$
$$\nabla^2\mathbf{A} = \mathbf{a}_\rho\left(\nabla^2 A_\rho - \frac{2}{\rho^2}\frac{\partial A_\phi}{\partial\phi} - \frac{A_\rho}{\rho^2}\right) + \mathbf{a}_\phi\left(\nabla^2 A_\phi + \frac{2}{\rho^2}\frac{\partial A_\rho}{\partial\phi} - \frac{A_\phi}{\rho^2}\right) + \mathbf{a}_z\nabla^2 A_z$$

Vector Operations – Spherical coordinates

$$\nabla \alpha = a_r \cdot \frac{\partial \alpha}{\partial r} + a_\theta \frac{1}{r} \frac{\partial \alpha}{\partial \theta} + a_\phi \cdot \frac{1}{r \sin\theta} \cdot \frac{\partial \alpha}{\partial \phi}$$

$$\nabla \cdot A = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta}\left(A_\theta \cdot \sin\theta\right) + \frac{1}{r \sin\theta} \frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times A = \frac{1}{r^2 \sin\theta} \begin{vmatrix} a_r & r . a_\theta & r \sin\theta . a_\phi \\ \partial/\partial r & \partial/\partial \theta & \partial/\partial \phi \\ A_r & r . A_\theta & r \sin\theta . A_\phi \end{vmatrix}$$

$$\nabla^2 \alpha = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \alpha}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta . \frac{\partial \alpha}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \alpha}{\partial \phi^2}$$

$$\nabla \times \nabla \times A = \nabla(\nabla \cdot A) - \nabla^2 A$$

$$\frac{d}{dx}(\sin ax) = a \cos x$$

$$\frac{d}{dx}(\cos ax) = -a \sin x$$

$$\int \sin ax . dx = -(\cos ax)/a$$

$$\int \cos ax . dx = \sin ax / a$$