MPZ 4230 – Engineering Mathematics II
Assignment No. 01 – Academic Year 2007

(01) (a) If the electric potential V at any point (x, y) is $V = \ln\sqrt{x^2 + y^2 + z^2}$. Find the rate of change of V at (3, 4, 5) in the direction towards point (2, 6, 7)

(b) Let $F(x, y, z) = (x^2y + xz)\underline{i} + \left(\frac{x^3}{3} - \cos z\right)\underline{j} + \left(\frac{x^2}{2} + y\sin z\right)\underline{k}$

Is this vector field is conservative

(c) Determine the Taylor polynomial of first order for the function $f(x, y) = e^{xy} + e^y$ at the origin.

(02) Show that the force $\underline{F}$ defined by $F = 3x^2y\underline{i} + (x^3 + 1)\underline{j} + 9z^2\underline{k}$ represents a conservative field of force. Find a scalar potential ϕ such that $F = \nabla\phi$. Hence find the work done in moving a particle of unit mass under this field of force from the point (0, 0, 0) to the point (1, 1,1).

(03) A torpedo has the shape of a cylinder with conical ends. For given surface area, show that the dimensions which give maximum volume are, $l = h = \frac{2r}{\sqrt{5}}$, where l is the length of the cylinder, r its radius and h the altitude of the cones.

(04) (a) Solve the equation $z^2 = 1 + i$

(b) If

$$c = 1 + \cos\theta + ------ + \cos(n-1)\theta$$
$$s = \sin\theta + ------ + \sin(n-1)\theta,$$

Prove that

$$c = \frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}\cos\frac{(n-1)\theta}{2} \quad \text{and}$$

$$s = \frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}\sin\frac{(n-1)\theta}{2}$$

If $\theta \neq 2k\pi, k \in Z$

(c) Find the modulus and principal argument of,

$$Z = \left(\frac{\sqrt{3} + i}{1 + i}\right)^{17}$$

and hence express z in modulus – argument form.

(05) (a) If $f(z) = u + iv$ analytic at $z = z_0$. Then show that u, v satisfy Laplace's equation at $z = z_0$

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} = 0$$

(b) Show that

$u = xe^x \cos y - ye^x \sin y$ function is harmonic.

(c) For above harmonic function find the conjugate harmonic function V and express $u + iv$ as an analytic function of z.

(d) Evaluate $\oint \bar{z} dz$ around the square at $z = 0, 2, 2 + 2i, 2i$.

(06) (a) Evaluate $\int_0^{1+i} (x + y) dz$ along each of the following paths.

(i) Along the y axis to i and then horizontally to $1 + i$.
(ii) Along the line $y = x$
(iii) Along the parabola $y = x^2$
(iv) Along the x axis to 1 and then vertically $1 + i$.

(b) (i) If c is a circle of radius r and center z_0 and if n is an integer, determine the value $\int_c \frac{dz}{(z - z_0)^{n+1}}$

(ii) Show that $\int_0^{2\pi} \cos^{2n} \theta d\theta = \int_0^{2\pi} \sin^{2n} \theta d\theta = \frac{1}{2^{2n}} \binom{2n}{n} 2\pi$

[Hint: Transform the integral by the substitution $z = e^{i\theta}$ then use the binomial theorem and the results of part (i)]

According to the Activity Diary due date please send your answers to the following address.

Course Co-ordinator-MPZ 4230
Dept. of Mathematics & Philosophy of Engineering
Faculty of Engineering Technology
The Open University of Sri Lanka
Nawala
Nugegoda

Model Answer 01 – MPZ 4230

Academic Year 2007

(01). (a). Let electric potential at any point = v(x, y)

$$v = \ln\sqrt{x^2+y^2+z^2} = \frac{1}{2}\ \ln(x^2+y^2+z^2)$$

$$\frac{\partial v}{\partial x}\Big|_{(3,4,5)} = \frac{1}{2}\frac{2x}{\left(x^2+y^2+z^2\right)} = \frac{x}{\left(x^2+y^2+z^2\right)} = \frac{3}{50}$$

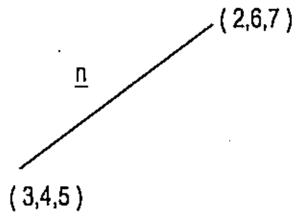
$$\frac{\partial v}{\partial y}\Big|_{(3,4,5)} = \frac{1}{2}\frac{2y}{\left(x^2+y^2+z^2\right)} = \frac{y}{\left(x^2+y^2+z^2\right)} = \frac{4}{50}$$

$$\frac{\partial v}{\partial z}\Big|_{(3,4,5)} = \frac{1}{2}\frac{2z}{\left(x^2+y^2+z^2\right)} = \frac{z}{\left(x^2+y^2+z^2\right)} = \frac{5}{50}$$

Then we find direction

Unit Vector $\underline{n} = -\frac{1}{3}\ \underline{i} + \frac{2}{3}\underline{j} + \frac{2}{3}\underline{k}$

$$= n_1\ \underline{i} + n_2\ \underline{j} + n_3\underline{k}$$



$$\nabla v \,.\, \underline{n} = \frac{\partial v}{\partial x}\ \underline{n_1} + \frac{\partial v}{\partial y}\ \underline{n_2} + \frac{\partial v}{\partial z}\ \underline{n_3}$$

$$= \frac{3}{50}\times -\frac{1}{3} + \frac{4}{50}\times\frac{2}{3} + \frac{5}{50}\times\frac{2}{3}$$

$$= \frac{-3+8+10}{150} = \frac{1}{10}$$

(b). $F = (x^2y + x\,z)\ \underline{i} + \left(\frac{x^3}{3} - \cos\ z\right)\underline{j} + \left(\frac{x^2}{2} + y\ \sin z\right)\ \underline{k}$

$$= F_1\ \underline{i} + F_2\ \underline{j}\ + F_3\ \underline{k}$$

$$\frac{\partial F_1}{\partial y} = x^2 \qquad \frac{\partial F_2}{\partial x} = x^2 \qquad \frac{\partial F_3}{\partial x} = x$$

$$\frac{\partial F_1}{\partial z} = x^2 \qquad \frac{\partial F_2}{\partial z} = \sin z \qquad \frac{\partial F_3}{\partial y} = \sin z$$

$$\text{curl } F = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\underline{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\underline{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\underline{k}$$

$$= (\sin z - \sin z)\ \underline{i} + (x - x)\ \underline{j} + (x^2 - x^2)\ \underline{k}$$

$$= 0$$

(c). $f(x, y) = e^{xy} + e^y$

$$\frac{\partial f}{\partial x} = ye^{xy} \qquad \frac{\partial f}{\partial y} = e^y + xe^y$$

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = 0 \qquad \left.\frac{\partial f}{\partial y}\right|_{(0,0)} = 1$$

$$p_1(x, y) = f(0, 0) + \left.\frac{\partial f}{\partial y}\right|_{(0,0)}(x - 0) + \left.\frac{\partial f}{\partial y}\right|_{(0,0)}(y - 0)$$

$$= 2 + y$$

(2).

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2y & x^3+1 & 9z^2 \end{vmatrix}$$

$$= (0 - 0)\ \underline{i}\ (0 - 0)\ \underline{j} + (3x^2 - 3x^2)\ \underline{k}$$

$$= 0$$

$\therefore$ F is conservative

$$F = \nabla\phi$$

$$3x^2y\ \underline{i} + (x^3 + 1)\ \underline{j} + 9z^2\ \underline{k} = \frac{\partial \phi}{\partial x}\ \underline{i} + \frac{\partial \phi}{\partial y}\ \underline{j} + \frac{\partial \phi}{\partial z}\underline{k}$$

$$\frac{\partial \phi}{\partial x} = 3x^2y \qquad \text{-------------} (1)$$

$$\frac{\partial\phi}{\partial y} = x^3 + 1 \quad \text{--------------- (2)}$$

$$\frac{\partial\phi}{\partial z} = 9z^2\, k \quad \text{--------------- (3)}$$

By (1) $\frac{\partial\phi}{\partial x} = 3x^2y$

$$\phi = \frac{3x^3y}{3} + f(y,z) \quad \text{------------ (A)}$$

$$\frac{\partial\phi}{\partial y} = x^3 + \frac{\partial f(y,z)}{\partial y}$$

By (2) $x^3 + 1 = x^3 + \frac{\partial f(y,z)}{\partial y}$

$$1 = \frac{\partial f(y,z)}{\partial y}$$

$$\therefore\ f(y, z) = y + g(z)$$

By (A) $-x^3y + \phi = y + g(z)$

$$\frac{\partial\phi}{\partial z} = \frac{\partial\ g(z)}{\partial z}$$

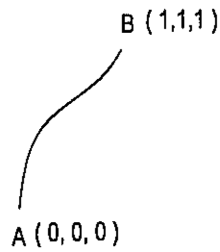
By (3) $\frac{\partial(g(z))}{\partial z} = 9z^2$

$$g(z) = \frac{9z^3}{3} = 3z^2 + c$$

$$\therefore \phi = x^3y + y + 3z^3 + c$$

$$\text{work done} = \int_A^B f.dr$$

$$= \int_A^B (3x^2y\underline{i} + (x^3+1)\underline{j} + 9z^2\underline{k}) . (dx\underline{i} + dy\underline{j} + dz\underline{k})$$



$$= \int_{\pi}^{B} 3x^2y\,dz + (x^3+1)\,dy + 9z^2\,dz$$

$$= \int_{A}^{B} d\,(yx^3+y) \int_{A}^{B} 9z^2\,dz$$

$$= \left[yx^3+y\right]_{(0,0,0)}^{(1,1,1)} + \left[3z^2\right]_{(0,0,0)}^{(1,1,1)}$$

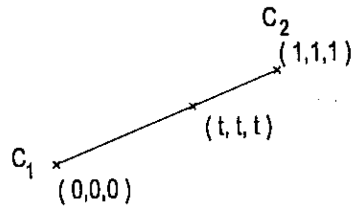
$$= (1+1)+3-0$$

$$= 5$$

method 2

$$r = x\underline{i} + y\underline{j} + z\underline{k}$$

$$r = t\underline{i} + t\underline{j} + t\underline{k}$$



$$\frac{dr}{dt} = \underline{i} + \underline{j} + \underline{k}$$

$$\int_{c_1}^{c_2} F.dr = \int_{c_1}^{c_2} F.d(x^2\underline{i} + y\underline{j} + z\underline{k})$$

$$= \int_{c_1}^{c_2} (3x^2y\ \underline{i} + (x^3+1)\ \underline{j} + 9z^2\underline{k})\ .\ (x\underline{i} + y\underline{j} + z\underline{k}).dt$$

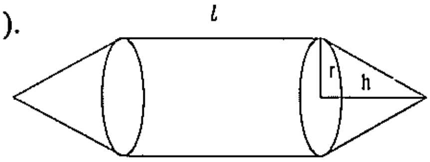
$$= \int_{0}^{1} 3t^3\ \underline{i} + (t^3+1)\underline{j} + 9z^2\underline{k}\ \ .\ (\underline{i} + \underline{j} + \underline{k})\ .\ dt$$

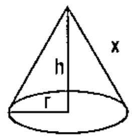
$$= \int_{0}^{1} 4t^3 + 9t^2 + 1\ \ dt$$

$$= \left[t^4 + 3t^3 + t\right]_0^1$$

$$= 5$$

(03).





$S = \pi r x$

$V = 1/3\ \pi r^2 h$

Surface area S

$$S = 2\pi r\ell + \pi r\,(r^2 + h^2)^{1/2} \times 2 \qquad \text{-------------} \ (1)$$

Volume V

$$V = \pi r^2\ell + \left(\frac{1}{3}\pi r^2 h\right) \times 2 \qquad \text{-------------} \ (2)$$

We have to find out maximum value of V subject to the condition given by (1)

Consider the function

$$F(r, \ell, h) = \pi r^2\ \ell + \frac{2}{3}\pi r^2\ \ell + \lambda\,[(2\pi r\ \ell + 2\pi r\ (r^2 + h^2)^{1/2} - S]$$

$$= \pi r^2\ (\ell + {}^{2h}\!/_{3}) + 2\lambda\pi r\,[\ \ell + (r^2 + h^2)^{1/2} - \lambda S]$$

The value of ℓ ,r ,h that make V stationary are given by,

$$\frac{\partial F}{\partial \ell} = 0 \Rightarrow \pi r^2 + 2\pi r\lambda = 0 \qquad [\because r \neq 0]$$

$$r + 2\lambda = 0 \quad \text{-------------} \ (3)$$

$$\lambda = -\frac{r}{2}$$

$$\frac{\partial F}{\partial r} = 0 \Rightarrow 2\pi r\ \left(\ell + {}^{2h}\!/_{3}\right) + 2\lambda\pi\ell + 2\lambda\pi.\frac{r}{2}.2r\left(r^2 + h^2\right)^{-1/2} + 2\left(r^2 + h^2\right)^{1/2}\lambda\pi = 0$$

$$2\lambda\ell + 2r\ (\ell\ +\ {}^{2h}\!/_{3}) + \frac{2\lambda r^2}{\left(r^2 + h^2\right)^{1/2}} + 2\left(r^2 + h^2\right)^{1/2}\lambda = 0 \ \text{-----------} \ (4)$$

$$\frac{\partial F}{\partial h} = 0 \Rightarrow \pi r^2\,\frac{2}{3} + 2\lambda\pi r\frac{1}{2}\ 2h\left(r^2 + h^2\right)^{-1/2} = 0$$

$$\frac{2r^2}{3}+\frac{2\lambda hr}{\left(r^2+h^2\right)^{1/2}}=0$$

$$\frac{r}{3}+\frac{\lambda h}{\left(r^2+h^2\right)^{1/2}}=0$$

$$r\sqrt{r^2+h^2}=3h\times-\frac{r}{2}$$

$$r^2+h^2=\frac{9h^2}{4}$$

$$r^2=\frac{5h^2}{4}$$

$$r=\frac{\sqrt{5}h}{2}$$

$$h=\frac{2r}{\sqrt{5}}$$

$$(4)\ \Rightarrow\ \ell r+\frac{4r^2}{3\sqrt{5}}-\frac{r\ell}{2}-\frac{r^3}{2\sqrt{r^2+\frac{4r^2}{5}}}-\frac{r}{2}\left(\frac{3r}{\sqrt{5}}\right)=0$$

$$\frac{\ell}{2}=\frac{3r}{2\sqrt{5}}+\frac{r}{2\sqrt{1+\frac{4}{5}}}-\frac{4r}{3\sqrt{5}}$$

$$=\frac{3r}{2\sqrt{5}}+\frac{\sqrt{5}r}{6}-\frac{4r}{3\sqrt{5}}$$

$$=\frac{r}{\sqrt{5}}\left(\frac{3}{2}+\frac{5}{6}-\frac{4}{3}\right)$$

$$\frac{\ell}{2}=\frac{r}{\sqrt{5}}$$

$$\ell=\frac{2r}{\sqrt{5}}$$

$$\ell=h=\frac{2r}{\sqrt{5}}$$

(4) (a). put $z = x + iy$

Then equation becomes

$(x + iy)^2 = x^2 - y^2 + 2xyi = 1 + i$

Comparing real & imaginary parts

$x^2 - y^2 = 1$

$2xy = 1$

Hens $x \neq 0$ and $y = \dfrac{b}{2x}$ consequently

$$x^2 - \left(\frac{1}{2x}\right)^2 = 1$$

So $4x^4 - 4x^2 - 1 = 0$

$$x^2 = \frac{4 \pm \sqrt{16+16}}{8} = \frac{1 \pm \sqrt{2}}{2}$$

Hence $x^2 = \dfrac{1 \pm \sqrt{2}}{5} \Rightarrow x = \pm\sqrt{\dfrac{1 \pm \sqrt{2}}{2}}$

Then $y = \dfrac{1}{2x} = \pm \dfrac{1}{\sqrt{2}\,\sqrt{1 \pm \sqrt{2}}}$

Hence the solutions are

$$Z = \pm\left(\sqrt{\frac{1 \pm \sqrt{2}}{2}} + \frac{i}{\sqrt{2}\sqrt{1 \pm \sqrt{2}}}\right)$$

(b). $c + is = 1 + (\cos\theta + i \sin \theta) + \ldots\ldots + (\cos (n - 1) + i \sin (n - 1)\,\theta)$

$= 1 + e^{i\theta} + \ldots\ldots\ldots\ldots + e^{i(n-1)\theta}$

$= 1 + z + \ldots\ldots + z^{n-1}$; where $z = e^{i\theta}$

$= \dfrac{1 + z^n}{1 - z}$; if $z \neq 1$ ..... i.e $\theta \neq 2k\pi$

$$= \frac{1 - e^{in\theta}}{1 - e^{i\theta}} = \frac{e^{\frac{in\theta}{2}}\left(e^{-\frac{in\theta}{2}} - e^{\frac{in\theta}{2}}\right)}{e^{\frac{i\theta}{2}}\left(e^{-\frac{i\theta}{2}} - e^{\frac{i\theta}{2}}\right)}$$

$$= e^{i(n-1)}\ \frac{\theta}{2}\quad \frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}$$

$$= [\cos (n-1)\frac{\theta}{2} + i \sin (n-1)\frac{\theta}{2}) \ \frac{\sin \frac{n\theta}{2}}{\sin \frac{\theta}{2}}$$

The result follows by equating real and imaginary parts

c) $|z| = \frac{|\sqrt{3}+i|^{17}}{|1+i|^{17}} = \frac{2^{17}}{(\sqrt{2})^{17}} = 2^{17/2}$

$$\text{Arg } z = 17 \text{ Arg} \left(\frac{\sqrt{3}+i}{1+i} \right)$$

$$= [\text{Arg} (\sqrt{3}+i) - \text{Arg} (1+i)]$$

$$= 17 \left(\frac{\pi}{6} - \frac{\pi}{4} \right) = \frac{-17\pi}{12}$$

Hence $\text{Arg } z = \left(\frac{-17\pi}{12} \right) + 2k\pi$, where k is an integer.

We see that k = 1 and hence $\text{Arg } z = \frac{7\pi}{12}$

Consequently $z = 2^{17/2} (\cos \frac{7\pi}{12} + i \sin \frac{7\pi}{12})$

(5). (a). If f(z) is analytic then

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ ---------- (1) $\qquad$ $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ ------------ (2)

(1). Differentiate with respect to x

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial y \, \partial x} \qquad \text{--------------- (3)}$$

(2). d.w.r. y $\frac{\partial^2 u}{\partial^2 y} = -\frac{\partial^2 v}{\partial x \, \partial y}$ --------------- (4)

(3) + (4) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Similarly $\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$

(b). $\frac{\partial u}{\partial x}=(1+x)e^x \cos y - ye^x \sin y$

$$\frac{\partial^2 u}{\partial x^2} = (2+x)\, e^x \cos y - ye^x \sin y \qquad \text{------------ (A)}$$

$$\frac{\partial u}{\partial y} = -xe^x \sin y - e^x (\sin y + y \cos y)$$

$$\frac{\partial^2 u}{\partial y^2} = -xe^x \cos y - e^x (2 \cos y - y \sin y) \text{ ---------------- (5)}$$

(A) & (B) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

(c). $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} = (1+x)\, e^x \cos y - ye^x \sin y$

$$\int dv = \int (1+x)\, e^x \cos y - ye^x \sin y \; dy$$

$$v = -e^x \int y \sin y \, dy + (1+x)\, e^x \sin y$$

$$v = -e^x [-y \cos y + \sin y] + (1+x)\, e^x \sin y + f(x) \text{ ---------- (A)}$$

$$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}=-xe^x \sin y - e^x(\sin y + y\cos y)$$

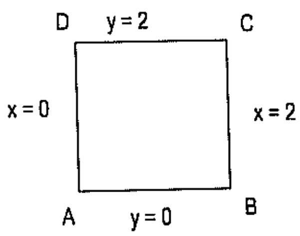
$$v = -\int xe^x \sin y \, dx \quad -\int e^x(\sin y + y \cos y)\, dx$$

$$v = (xe^x - e^x) \sin y + e^x [\sin y + y \cos y] + f(y) \qquad \text{-------- (B)}$$

Comparing (A) & (B)

$$v = xe^x \sin y + ye^x \cos y + c$$

(d).



Let $z = x + iy$

$\bar{z} = x - iy = u + iv$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0 \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\bar{z}$ is analytic

$$\oint \bar{z}\ dz = \int_{AB} \bar{z}\,dz + \int_{BC} \bar{z}\,dz + \int_{CD} \bar{z}\,dz + \int_{DA} \bar{z}\,dz$$

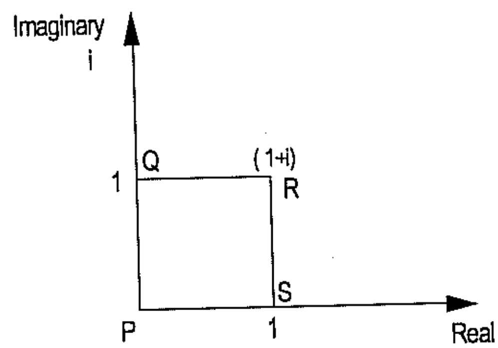
$$= \int_0^z x\ dx + \int_0^2 (2 - iy)i\ dy + \int_2^0 (x - 2i)dx + \int_2^0 -iy\ i\ dy$$

$$= \left[\frac{x^2}{2}\right]_0^2 + i\left[2y - \frac{iy^2}{2}\right]_0^2 + \left[\frac{x^2}{2} - 2xi\right]_2^0 + i\left[-\frac{iy^2}{2}\right]_2^0$$

$= 2 + (4i - 2i) + (-2 + 4i) + 2i$

$= 8i$

(06). (a). (i).



$z = x + iy$

$dz = dx + i\ dy$

$$\int_0^{1+i} (x + y)dz = \int_{PQ} (x + y)dz + \int_{QR} (x + y)dz$$

$$= \int_0^1 y\ i dy + \int_0^1 (x + 1)dx$$

$$= i\left[\frac{y^2}{2}\right]_0^1 + \left[\frac{x^2}{2} + x\right]_0^1$$

$$= \frac{1}{2}i + \frac{3}{2}$$

(ii). z = x + iy

z = x + ix

z = x(1 + i)

dz = (1 + i) dx

$$\int_0^{1+i} (x+y)dz = \int_0^1 (x+x)(1+i)dx$$

$$= \int_0^1 2x(1+i)dx$$

$$= (1+i)\left[\frac{2x^2}{2}\right]_0^1$$

$$= (1+i)(1)^2$$

$$= (1+i)$$

(iii). z = x + iy

$z = x + ix^2$

$$\frac{dz}{dx} = (1 + 2xi)\,dx$$

$$\int_0^{1+i} (x+y) = \int_0^1 (x + x^2)(1 + 2xi)\,dx$$

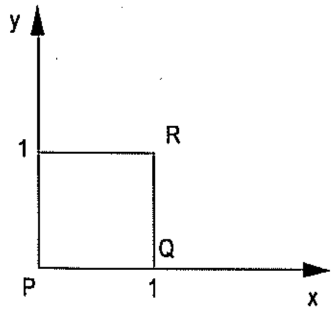
$$= \int_0^1 (x + 2xi + x^2 + 2x^3\ i)\,dx$$

$$= \left[\frac{x^2}{2} + \frac{2ix^3}{3} + \frac{x^3}{3} + \frac{2ix^4}{4}\right]_0^1$$

$$= \left[\frac{1}{2} + \frac{2i}{3} + \frac{1}{3} + \frac{i}{2}\right]$$

$$= \frac{5}{6} + \frac{7i}{6}$$

(iv). $\int_0^{1+i} (x+y)dz = \int_0^1 x\,dx + \int_0^1 (1+y)i\,dy$



$$= \left[\frac{x^2}{2}\right]_0^1 + \left[y + \frac{y^2}{2}\right]_0^1 i$$

$$= \frac{1}{2} + i\left[1 + \frac{1}{2}\right]$$

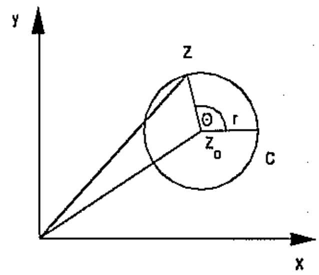
$$= \frac{1}{2} + \frac{3}{2}i$$

(06). (b).

(i). For convenience, let us make the substitution $z - z_0 = re^{i\theta}$ noting that the parameter θ varies from 0 to 2π as z varies around the circle in the counter clockwise direction.

Then $dz = rie^{i\theta}\,d\theta$; and the integral becomes $\int_0^{2\pi} \frac{rie^{i\theta}d\theta}{r^{n+1}e^{i(n+1)\theta}} = \frac{i}{r^n}\int_0^{2\pi} e^{-in\theta}d\theta$

If $n = 0$ this reduce to



$$i\int_0^{2\pi} d\theta = 2\pi i$$

If $n \neq 0$ $\frac{i}{r^n}\int_0^{2\pi} (\cos n\theta - i\sin n\theta)\,d\theta = 0$

(ii) let $z = e^{i\theta}$

$dz = ie^{i\theta}\,d\theta$

$dz = iz\,d\theta$

$d\theta = \frac{1}{iz}\,dz$ ----------- (1)

Then $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$

$$= \frac{e^{i\theta} + 1/e^{i\theta}}{2}$$

$$= \frac{z + 1/z}{2} \quad \text{---------------------(2)}$$

(1) & (2) substitute

$$\int_0^{2\pi} \cos^{2n}\theta \, d\theta = \int_0^{2\pi} \frac{\left(z+\frac{1}{z}\right)^{2n}}{2^{2n}} \frac{dz}{iz}$$

$$= \int_0^{2\pi} \frac{1}{2^{2n} iz}\left(z+\frac{1}{z}\right)^{2n} dz$$

$$= \frac{1}{2^{2n} i}\int_0^{2\pi} \frac{1}{z}\left\{z^{2n} + {}^{2n}c_1\, z^{2n-1} + \ldots\ldots + {}^{2n}c_k\ \ z^{2n-k} + \ldots\ldots \left(\frac{1}{z}\right)^{2n}\right\} dz$$

$$= \frac{1}{2^{2n} i}\int_0^{2\pi} z^{2n-1} + {}^{2n}c_1 z^{2n-2} + \ldots\ldots\ldots + {}^{2n}c_k z^{2n-k-1} + \ldots\ldots + \left(\frac{1}{z}\right)^{2n-1} dz$$

Using part I

$$= \frac{1}{2^{2n} i}\ 2\pi i \binom{2n}{n}$$

$$= \frac{i}{z^{2n}}\binom{2n}{n} \times 2\pi$$

Similarly

$$\int_0^{2\pi} \sin^{2n}\theta \, d\theta = \frac{1}{z^{2n}}\binom{2n}{n} \times 2\pi$$

$$\therefore \quad \int_0^{2\pi} \cos^{2n}\theta \, d\theta = \int_0^{2\pi} \sin^{2n}\ \theta \, d\theta = \frac{1}{z^{2n}}\binom{2n}{n} 2\pi$$