1. (a). Let M denote the set of ordered triples (x, y, z) of real numbers with the operations of addition & multiplications

$(x, y, z) + (x^1, y^1, z^1) = (x + z^1, y + y^1, z + z^1)$

$c \times (x, y, z) = (2c, cy, cz)$

Is M vector space? Why?

(b). Consider the vector space R^4. Is the subset S of vectors of the form (x_1, x_2, x_3, x_4). Where x_1, x_2 and x_3 are arbitrary and $x_4 \le 0$ a sub space? Why?

(c). Consider the vector space P_2 of polynomials of degree ≤ 2. Is the subset S of polynomials of the form

$p(t) = a_0 + a_1t + (a_0 + a_1)t^2$ a sub space

2. (a). Answer true or false, & prove

i). Span $(\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4) = R^3$, where

$$\underline{v}_1 = \begin{pmatrix}1\\2\\3\end{pmatrix} \quad v_2 = \begin{pmatrix}3\\2\\1\end{pmatrix}, \quad v_3 = \begin{pmatrix}1\\0\\-1\end{pmatrix}; \quad v_4 = \begin{pmatrix}0\\1\\1\end{pmatrix}$$

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ii). The four vectors in (i) are independent

iii). In part (i), all vectors, $\underline{v} = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$ in span $(\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4)$ satisfy a linear equation

$ax_1 + bx_2 + cx_3 = 0$ for scalars a, b, c not all zero

iv). The rank of the matrix $\begin{pmatrix}1 & 2 & 3\\3 & 2 & 1\\1 & 0 & -1\\0 & 1 & 1\end{pmatrix}$ is 3

(b). Consider the following two vectors in R^3

$$v_1 = \begin{bmatrix}1\\3\\2\end{bmatrix} \text{ and } v_2 = \begin{bmatrix}-1\\4\\3\end{bmatrix}$$

i). Find the third none zero vector v_3 so that the set v_1, v_2, v_3 are linearly dependent. (explain)

ii). Find a third vector v_3 so that the set v_1, v_2, v_3 are linearly independent. (explain)

3. (a). T is a linear transformation of R^3 in to R^2 such that

$$T\begin{bmatrix}1\\-1\\2\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}, \quad T\begin{bmatrix}2\\1\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}, \quad T\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix}$$

i). Is T 1 – 1?

ii). Determine the matrix of T relative to the standard bases in R^3 and R^2

(b). Let W be the following subspace of R^3

$$W = \text{comb}\left(\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}3\\3\\-3\end{bmatrix}\right)$$

i). Show that $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}$ are basis of W

for ii) and iii) below, let T be the following linear transformation $T: W \to R^3$

$$T\left(\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}\right) = \begin{bmatrix}1 & 0 & -1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}$$

for those $w = \begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}$ in R^3 with belong to W

ii). What is the dimension of Range (T)?
iii). What is the dimension of Ker (T)?

4. (a). The vectors $u_1 = (1, 2, 1, 0)$, $u2 = (3, 3, 3, 0)$, $u_3 = (2, -10, 0, 0)$, $u_4 = (-2, 1, -6, 2)$ are linearly independent in R^4. Consider R^4 as a real inner product space with the Euclidean inner product, and apply the Gram-Schmidt process to this basis.

(b). Using above vectors verify the theorem [theorem: if $\underline{u}$ and $\underline{v}$ are orthogonal vectors in an inner product space, then $\|\underline{u}+\underline{v}\|^2 = \|\underline{u}\|^2 + \|\underline{v}\|^2$

5. Prove that eigen values of a 3 × 3 diagonal matrix are the diagonal elements of the matrix. A matrix is given as $A = \begin{bmatrix}3 & 2 & 4\\2 & 0 & 2\\4 & 2 & 3\end{bmatrix}$

Find

a). Eigen value of A
b). Eigen vector of A
c). Obtain a matrix P such that $P^{-1}AP$ is a diagonal matrix.
d). If $S = P^{-1}AP$, state the special property of S and find S^{-1}
e). Using above results reduce the quadratic form Q(x) to form Q(y), where
$Q(x) = 3x^2_1 + 3x^2_3 + 4x_1x_2 + 4x_2x_3 + 8x_1x_3$
and $Q(y) = a_1 y^2_1 + a_2 y^2_2 + a_3 y^2_3$

f). Obtain the relationship between $x = (x_1, x_2, x_3)'$ and $y = (y_1, y_2, y_3)'$

Model Answer 04 – MPZ 4230
Academic Year 2007

(01). (a). Axiom [A 9]

If $a, b, \in C$, $\underline{u} \in \underline{V}$

Then $\underline{u}(a+b) = a\underline{u} + b\underline{u}$

Check the axiom [A9] for

$a = 1, b= 2, \underline{u} = (1, 2, 3)$

L.H.S. = $\underline{u}\ (a+b) = 3(1, 2, 3)$

$= (6, 6, 9)$ $[\because\ c(x, y, z) = (2c, cy, cz)]$

R.H.S. $a\underline{u} + b\underline{u} = 1(1,2,3) + 2(1,2,3)$

$= (2, 2, 3) + (4, 4, 6)$

$= (8, 6,9)$

L.H.S. $\neq$ R.H.S.

$\therefore$ Axiom [A9] contradict

$\therefore$ This is not a vector space

b). $\underline{w} = \{x, y, z, t\ ;\ t < 0\}$

we want to prove that

If k is any scalar and $\underline{u}$ is any vector then $k\underline{u} \in \underline{w}$

Let $k = -1$ $\underline{u} = (1, 2, 0, -1)$

$k\underline{u} = (-1, -2, 0 , 1) \notin w$

$\therefore$ w is not a sub space

c)

We what to prove

$u.\ w \in w \Rightarrow \propto u + \beta \underline{w} \in w$ for every $\propto, \beta \in k$

let $u = p(t_1) = t_1 + a_0 + a_1 t_1 + (a_0 + a_1) t_1^2$

$v = p(t_2) = t_2 + a_0 + a_1 t_2 + (a_0 + a_1) t_2^2$

$\propto \underline{u} + \beta \underline{v} = \propto (t_1 + a_0 + a_1 t_1 + (a_0 + a_1) t_1^2) + p(t_2 + a_0 + a_1 t_2^2 + (a_0 + a_1) t_2^2)$

$= (\propto t_1 + \beta t_2) + a_0 (\propto + \beta) + a_1 (\propto t + \beta t_2) + (a_0 + a_1)\ (\propto t_1^2 + \propto\beta t_2^2)$

Degree of this polynomial is ≤ 2

$\therefore$ p(t) , is a subspace of p_2

(2). a). i).

False

Let $a_1 (1, 2, 3) + a_2 (3, 2, 1) + a_3 (1, 0, -1) + a_4 (0, 1, 1) = (x, y, z)$

Then

$a_1 + 3a_2 + a_3 = x$

$2a_1 + 2a_2 + a_4 = y$

$3a_1 + a_2 - a_3 + a_4 = z$

using this 3 equation, we can't find a_1, a_2, a_3, a_4 four variables
we can't span (v_1, v_2, v_3, v_4)

(ii). false

Let $a_1 (1, 2, 3) + a_2 (3, 2, 1) + a_3 (1, 0, -1) + a_4 (0, 1, 1) = 0$

Then

$a_1 + 3a_2 + a_3 = 0$

$2a_1 + 2a_2 + a_4 = 0$

$3a_1 + a_2 - a_3 + a_4 = 0$

Take one of "a" = t, then find other "a" s

When t vary

a_1, a_2, a_3, a_4 have infinite solution

$\therefore$ given vectors are not independent

(iii). false

$ax_1 + bx_2 + cx_3 = 0$ means

$$\begin{aligned} a+2b+3c&=0\\ 3a+2b+c&=0\\ a-c&=0\\ b+c&=0 \end{aligned} \Rightarrow \begin{pmatrix} 1 & 2 & 3\\ 3 & 2 & 1\\ 1 & 0 & -1\\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} a\\ b\\ c \end{pmatrix}=0$$

$$\begin{pmatrix} 1 & 2 & 3\\ 3 & 2 & 1\\ 1 & 0 & -1\\ 0 & 1 & 1 \end{pmatrix} \underset{\Rightarrow}{R_3 = R_1 - R_3} \begin{pmatrix} 1 & 2 & 3\\ 3 & 2 & 1\\ 0 & 2 & 4\\ 0 & 1 & 1 \end{pmatrix} \begin{matrix} R_4 = R_3 - 2R_4\\ \Rightarrow\\ R_2 = 3R_1 - R_2 \end{matrix} \begin{pmatrix} 1 & 2 & 3\\ 0 & 4 & 8\\ 0 & 2 & 4\\ 0 & 0 & 2 \end{pmatrix} \underset{\Rightarrow}{R_2 = R_2 - 2R_3} \begin{pmatrix} 1 & 2 & 3\\ 0 & 0 & 0\\ 0 & 2 & 4\\ 0 & 0 & 2 \end{pmatrix}$$

$$\text{Then} \begin{pmatrix} 1 & 2 & 3\\ 0 & 0 & 0\\ 0 & 2 & 4\\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} a\\ b\\ c \end{pmatrix}=\begin{pmatrix} 0\\ 0\\ 0 \end{pmatrix}$$

$\therefore c = 0, \ b = 0, \ a = 0$

Haven't non zero solution for a, b, c

(iv). True

$$A = \begin{pmatrix} 1 & 2 & 3\\ 3 & 2 & 1\\ 1 & 0 & -1\\ 0 & 1 & 1 \end{pmatrix} \begin{matrix} R_2 = 3R_1 - R_2\\ \Rightarrow\\ R_3 = R_1 - R_3 \end{matrix} \begin{pmatrix} 1 & 2 & 3\\ 0 & 4 & 8\\ 0 & 2 & 4\\ 0 & 1 & 1 \end{pmatrix} \begin{matrix} R_2 = 2R_3 - R_2\\ \Rightarrow\\ R_4 = R_3 - 2R_4 \end{matrix} \begin{pmatrix} 1 & 2 & 3\\ 0 & 0 & 0\\ 0 & 2 & 4\\ 0 & 0 & 2 \end{pmatrix}$$

We can find only one zero value row. Other 3 are non zero value row

$\therefore$ rant of A = 3

(b). (i). Take $\propto v_1 + \beta v_2 + \gamma v_3 = 0$ $(\propto, \beta, \gamma, \neq 0)$

$$\frac{\alpha}{\gamma} v_1 + \frac{\beta}{\gamma} v_2 + v_3 = 0$$

$$v_3 = -\frac{\alpha}{\gamma} v_1 - \frac{\beta}{\gamma} v_2$$

$$= \lambda v_1 + \mu v_2$$

Taking any none zero value for λ and μ we can find none zero v_3 vector such that v_1, v_2, v_3 are linearly dependent.

(ii). let $v_1 = (x, y, z)$, $v_2 = (1, 3, 2)$, $0_3 = (-1, 4, -3)$
If v_1, v_3, v_3 are lineally independent.

$\propto v_1 + \beta v_2 + \gamma v_3 = 0$ [where $\propto = \beta = \gamma = 0$]

$$\begin{pmatrix} x & y & z \\ 1 & 3 & 2 \\ -1 & 4 & -3 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = 0$$

If v_1, v_2, v_3 are independent then $A = \begin{vmatrix} x & y & z \\ 1 & 3 & 2 \\ -1 & 4 & -3 \end{vmatrix} \neq 0$

Then we want to find x, y, z such that $|A| \neq 0$

Ex: Take $v_3 = (1, 4, 1)$

$$A = \begin{pmatrix} 1 & 4 & 1 \\ 1 & 3 & 2 \\ -1 & 4 & 3 \end{pmatrix} \begin{matrix} R_3 = R_2 + R_3 \\ \Rightarrow \\ R_2 = R_1 - R_2 \end{matrix} \begin{pmatrix} 1 & 4 & 1 \\ 0 & 1 & -1 \\ 0 & 7 & 5 \end{pmatrix} \begin{matrix} R_3 - R_3 - 7R_2 \\ \Rightarrow \end{matrix} \begin{pmatrix} 1 & 4 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 12 \end{pmatrix}$$

$A \neq 0$

$\therefore$ v_3 is a vector, such that v_1, v_2, v_3 are linearly independent

(3). (a). We first express $\underline{x} = (x_1, x_2, x_3)$ as a linear combination of $v_1 = (1, -1, 2)$, $v_2 = (2, 1, 0)$, $v_3 = (1, 1, 1)$

$T(k_1 v_1 + k_2 v_2 + k_3 v_3) = k_1 T(v_1) + k_2 T(v_2) + k_3 T(v_3)$

$(x_1, x_2, x_3) = k_1 (1, -1, 2) + k_2 (2, 1, 0) + k_3 (1, 1, 1)$ ---------- (A)

Comparing

$x_1 = k_1 + 2k_2 + k_3$ --------------- (1)

$x_2 = -k_1 + k_2 + k_3$ --------------- (2)

$x_3 = 2k_1 + k_3$ --------------- (3)

$(1) - 2 \times (2)$

$$x_1 - 2x_2 = 3k_1 - k_3$$

$(3) + (4)$ $x_1 - 2x_2 + x_3 = 5k_1$

$$K_1 = \frac{1}{5}(x_1 - 2x_2 + x_3)$$

$$K_3 = \frac{1}{5}\,(-2x_1 + 4x_2 + 3x_3)$$

$$K_2 = \frac{1}{5}\,(3x_1 - x_2 - 2x_3)$$

by (A)

$$(x_1, x_2, x_3) = \frac{1}{5}\,(x_1 - 2x_2 + x_3)\,(1, -1, 2) + \frac{1}{5}\,(3x, -x_2 - 2x_3)\,(2, 1, 0) + \frac{1}{5}\,(-2x_1 + 4x_2 + 3x_3)\,(1, 1, 1)$$

Thus

$$T\,(x_1, x_2, x_3) = \frac{1}{5}\,(x_1 - 2x_2 + x_3)\,T\,(1, 1, 1) + \frac{1}{5}\,(3x_1 - x_2 - 2x_3)\,T\,(2, 1, 0) + \frac{1}{5}\,(-2x_1 + 4x_2 + 3x_3)\,T\,(1, 1\,.\,0)$$

$$= \frac{1}{5}\,(x_1 - 2x_2 + x_3)\,(2, 1) + \frac{1}{5}\,(3x_1 - x_2 - 2x_3)\,(1, 0) + \frac{1}{5}\,(-2x + 4x_2 + 3x_3)(1, -1)$$

$$= \frac{1}{5}\,[(2x_1 - 4x_2 + 2x_3 + 3x_1 - x_2 - 2x_3 - 2x_1 + 4x_2 + 3x_2),\; x_1 - 2x_2 + x_3 + 2x_2 - 4x_2 - 3x_3)]$$

$$= \frac{1}{5}\,[3x_1 - x_2 + 3x_3,\, 3x_1 - 6x_2 - 2x_3]$$

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \dfrac{3x_1 - x_2 + 3x_3}{5} \\ \dfrac{3x_1 - 6x_2 - 2x_3}{5} \end{pmatrix}$$

(i). Let $x, y \in R^3$

$x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$

$T(x) = T(x_1, x_2, x_2)$, $T(y) = T(y_1, y_2, y_3)$

If $T(x) = T(y)$

Then $\begin{pmatrix} 3x_1 - x_2 + 3x_3 \\ 3x_1 - 6x_2 - 2x_3 \end{pmatrix} = \begin{pmatrix} 3y_1 - y_2 + 3y_3 \\ 3y_1 - 6y_2 - 2y_3 \end{pmatrix}$

$x_1 = y$, $x_2 = y_2$ & $x_3 = y_3$

$\therefore$ T is one to one

(ii).

$$T = \begin{pmatrix} \frac{3}{5} & \frac{-1}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{-6}{5} & \frac{-2}{5} \end{pmatrix}$$

(b) (i). $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 2 & 1 & 0 \\ 3 & 3 & -3 \end{pmatrix} \begin{matrix} R_4 = R_1 - 3R_2 \\ \Rightarrow \\ R_3 = R_1 + R_2 \end{matrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\therefore$ (1, 0, 1) , (1, 1, -1) are basis of w.

(ii). $T\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} w_1 - w_3 \\ 0 \\ 0 \end{pmatrix} = (w_1 - w_3)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Dimension of Range (T) = 1

If $= T\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = 0$ Then $\ker T = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$

$= \Rightarrow \begin{pmatrix} w_1 - w_2 \\ 0 \\ 0 \end{pmatrix} = 0 \Rightarrow w_1 = w_2 \ \Rightarrow \ker T = \begin{pmatrix} w_1 \\ w_2 \\ w_1 \end{pmatrix}$

(rank of T) + (nulity of T) = n

1 + nulity of T = 3

dimension of ker (T) = nul to of T = 2

4. Step 1

$v_1 = \dfrac{u_1}{\|u_1\|} = \dfrac{(1,2,1,0)}{\sqrt{6}} = \left(\dfrac{1}{\sqrt{6}}, \dfrac{2}{\sqrt{6}}, \dfrac{1}{\sqrt{6}}, 0\right)$

step 2

$v_2 = \dfrac{\underline{u}_2 - \text{proj } w_1 \underline{u}_2}{\|\underline{u}_2 - \text{proj } w_1 u_2\|}$

$u_2 - \text{proj } w_1 u_2 = u_2 - (u_2 \ . \ v_1)v_1$

$u_2 - \text{proj } w_1 u_2 = (3, 3, 3, 0) - 2\sqrt{6}\left(\dfrac{1}{\sqrt{6}}, \dfrac{2}{\sqrt{6}}, \dfrac{1}{\sqrt{6}}, 0\right)$

$= (1, -1, 1, 0)$

$v_2 = \dfrac{1}{\sqrt{3}}\ (1, -1, 1, 0)$

step 3

$v_3 = \dfrac{u_3 - \text{proj } w_1 u_3}{\|u_3 - \text{proj } w_1 u_3\|}$

$u_3 - \text{proj } w_1 u_3 = u_2 - (u_3 \, . \, v_1)\, v_1 - (u_3 \, . \, v_2)\, v_2$

$= (2, -10, 0, 0) \quad - \dfrac{18}{\sqrt{6}}\left(\dfrac{1}{\sqrt{6}}, \dfrac{2}{\sqrt{6}}, \dfrac{1}{\sqrt{6}}, 0\right) - \dfrac{12}{\sqrt{3}}\left(\dfrac{1}{\sqrt{3}}, \dfrac{-1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, 0\right)$

$= (2, -10, 0, 0) + (3, 6, 3, 0) - 2(4, -4, 4, 0)$

$= (1, 0, -1, 0)$

Step 4

$$v_4 = \frac{u_4 - \text{proj } w_1 u_3}{\|u_4 - \text{proj } w_1 u_3\|}$$

u_4 - proj w_1 u_3 = $u_4 - (u_4 \cdot v_1)\, v_1 - (u_4 \cdot v_2)\, v_2 - (u_4 \cdot v_3)\, v_3$

= (-2, 1, -6, 2) + (1, 2, 1, 0) + (3, -3, 3, 0) + (-2, 0, 2, 0)

= (0, 0, 0, 2)

$$v_4 = \frac{(0,0,0,2)}{2} = (0,0,0,1)$$

ii. Take orthogonal vectors v_1 & v_2

$$\|v_1 + v_2\| = \left[\left(\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{6}}\right)^2 + \left(\frac{2}{\sqrt{6}} - \frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{3}}\right)^2\right]^{1/2} = 2$$

$\|v_1\| = 1 \quad \|v_2\| = 1$

$\therefore \|v_1 + v_2\| = \|v_1\| + \|v_2\|$

v_1, v_2 , v_3 , v_4 are orthogonal vectors. Taking any 2 vectors of v_1 , v_2 , v_3 , v_4 we can prove the given theorem.

5. (a). $A - \lambda I = \begin{bmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{bmatrix}$

$$3 - \lambda\, [-\lambda(3-\lambda)-4] + 2[8 - 2(3 - \lambda)] + 4\,[4 + 4\lambda] = 0$$
$$(3 - \lambda)\,[\lambda^2 - 3\lambda - 4] + 2(2\lambda + 2) + 16\,(1 + \lambda) = 0$$
$$3\lambda^2 - 9\lambda - 12 + \lambda^3 + 3\lambda^2 + 4\lambda + 4\lambda + 4 + 16 + 16\lambda = 0$$
$$-\lambda^3 + 6\lambda^2 + 15\lambda + 8 = 0$$
$$(\lambda + 1)\,(-\lambda^2 + 7\lambda + 8) = 0$$
$$(\lambda + 1)\,(-\lambda + 8)\,(\lambda + 1) = 0$$

$\lambda = -1$ and $\lambda = 8$

eigon values are -1, 8

(b). when $\lambda = -1$

$$\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = -1 \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$4a + 2b + 4c = 0$

$2a + b + 2c = 0$

$4a + 2b + 4c = 0$

Let $a = t_1$ & $c = t_2$

Then $b = -2t_1 - 2t_2$

$$\begin{pmatrix} t_1 \\ -2t_1-2t_2 \\ t_2 \end{pmatrix} = t_1\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + t_2\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} = tv_1 + tv_2$$

v_1 & v_2 are eigon vectors.

Let $\lambda = 8$

$-5a + 2b + 4c = 0$

$2a - 8b + 2c = 0$

$4a + 2b - 5c = 0$

Let $a = t_3$

Then $b = \frac{1}{2}\ t_3$

$c = t_3$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = t\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = tv_3$$

eigon vector is v_3

(c).

$$\text{let } p = \begin{pmatrix} 1 & 0 & 2 \\ -2 & -2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$p^{-1} = \frac{1}{-3}\begin{pmatrix} -5 & 2 & 4 \\ 4 & 2 & -5 \\ -2 & -1 & -2 \end{pmatrix}$$

$$p^{-1}AP = -\frac{1}{3}\begin{pmatrix} -5 & 2 & 4 \\ 4 & 2 & -5 \\ -2 & -1 & -2 \end{pmatrix}\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ -2 & -2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$= -\frac{1}{3}\begin{pmatrix} 5 & -2 & -4 \\ -4 & -2 & 5 \\ -16 & -8 & -16 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ -2 & -2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$= -\frac{1}{3}\begin{pmatrix} 9 & 0 & 9 \\ 0 & 9 & 0 \\ 0 & 0 & -72 \end{pmatrix} = 3\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

(d). $S.S = P^{-1}\underbrace{AP\ .\ p^{-1}}\ AP$

$S^2 = P^{-1}AI\,AP$

$S^2 = P^{-1} A^2 P$
$S^3 = P^{-1} A^2 P . P^{-1} AP$
$S^3 = P^{-1} A^3 P$

Similarly;
$S^N = P^{-1} A^N P$

also
$S = P^{-1} A P$
$S . S^{-1} = P^{-1} APS^{-1}$
$I = P^{-1} APS^{-1}$
$PI = Aps^{-1}$
$A^{-1}p = ps^{-1}$
$p^{-1} A^{-1} p = s^{-1}$

(e). $Q(x) = 3x^2_1 + 3x^2_3 + 4x_1x_2 + 4x_2x_3 + 8x_1x_3$
$= 3x^2_1 + ox^2_2 + 3x^2_3 + 4x_1x_2 + 4x_2x_3 + 8x_1x_3$

$$\therefore \text{ Given quadratic form } Q(x) = [x_1\ x_2\ x_3] \begin{bmatrix} 3x_1 & 2x_2 & 4x_3 \\ 2x_1 & 0x_2 & 2x_3 \\ 4x_1 & 2x_2 & 3x_3 \end{bmatrix}$$

$$= [x_1\ x_2\ x_3] \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

$I A I = A$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

$R_1 \to R_1 - 2R_2$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 0 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

$C_1 \to C_1 - 2C_2$

$$\begin{bmatrix}1 & -2 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 0 & 0\\-2 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}-5 & 2 & 0\\2 & 0 & 2\\0 & 2 & 3\end{bmatrix}$$

$C_2 \to C_2 + 2C_1/5$

$$\begin{bmatrix}1 & -2 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 2/5 & 0\\-2 & 1/5 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}-5 & 0 & 0\\2 & 4/5 & 2\\0 & 2 & 3\end{bmatrix}$$

$R_2 \to R_2 + 2\, R_1/5$

$$\begin{bmatrix}1 & -2 & 0\\2/5 & 1/5 & 0\\0 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 2/5 & 0\\-2 & 1/5 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}-5 & 0 & 0\\0 & 4/5 & 2\\0 & 2 & 3\end{bmatrix}$$

$C_3 \to C_3 - 5/2\, C_2$

$$\begin{bmatrix}1 & -2 & 0\\2/5 & 1/5 & 0\\0 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 2/5 & -1\\-2 & 1/5 & -1/2\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}-5 & 0 & 0\\0 & 4/5 & 0\\0 & 2 & -2\end{bmatrix}$$

$R_3 \to R_3 - 5/2\, R_2$

$$\begin{bmatrix}1 & -2 & 0\\2/5 & 1/5 & 0\\-1 & -1/2 & 1\end{bmatrix} A \begin{bmatrix}1 & 2/5 & -1\\-2 & 1/5 & -1/2\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}-5 & 0 & 0\\0 & 4/5 & 0\\0 & 0 & -2\end{bmatrix}$$

$$\therefore\ p^T A p = \begin{bmatrix}-5 & 0 & 0\\0 & 4/5 & 0\\0 & 0 & -2\end{bmatrix}$$

$$P = \begin{bmatrix}1 & 2/5 & -1\\-2 & 1/5 & -1/2\\0 & 0 & 1\end{bmatrix}$$

Thus the matrix a is reduced to the diagonal from B. The canonical form is

$Q = y^T B y$

$$= (y, y_2, y_3) \begin{bmatrix} -5 & 0 & 0 \\ 0 & 4/5 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= (-5y_1, \tfrac{4}{5} y_2, -2y_3) \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= -5y^5{}_1 + \tfrac{4}{5} y^2{}_2 - 2y^2{}_3$$

$$= a_1 y^2{}_1 + a_2 y^2{}_2 + a_3 y^2{}_2$$

$A_1 = -5$, $a_2 = \tfrac{4}{5}$, $a_3 = -2$