MPZ 4230 - Assignment No. 01 – Academic Year 2008

01. (a). If $z = f(t)$ and $t = \frac{x+y}{xy}$ show that $x^2 \frac{\partial z}{\partial x} = y^2 \frac{\partial z}{\partial y}$

(b). The two sides forming the right angle of a triangle are denoted by a and b. The hypotenuse is h. If there are possible errors of $\pm 0.5\%$ in measuring a and b, find the maximum possible error in calculating

(i). The area of the triangle

(ii). The length of h

02. (a). A rectangular box, open at the top, is to have a volume of 32 cm^3. What are the dimension of the box so that the surface area is a minimum?

(b). A jewel box is to be constructed of material that cost \$1 per square inch for the bottom, \$2 per square inch for the sides, and \$5 per square inch for the top. If the total volume is to be 96 cm^3, W dimensions will minimize the total cost of construction?

03. Let $F(x, y, z) = (x + 2y + 4z)\,\underline{i} + (2x - 3y - z)\,\underline{j} + (4x - y + 2z)\,\underline{k}$

(i). Is this vector field conservative?

(ii). If so, find ϕ such that $F = \nabla\phi$

04. (i). Find the square root of $8 + 4\sqrt{5}\,i$

(ii). Show that $|\alpha + \beta|^2 + |\alpha - \beta|^2 = 2\{|\alpha|^2 + |\beta|^2\}$; α and β being any complex number

(iii). Show that $\arg z + \arg \bar{z} = 2n\pi$

05. Let $u(x, y) = x^3 - 3xy^2 - 3x^2 + 3y^2$, where $(x, y) \in \Re^2$

(i). Show that u is a harmonic function

(ii). Find a function v such that $f(z) = u + iv$ is analytic

(iii). Express f(z) in terms of z

06. (i). State the Cauchy's Residue Theorem

(ii). By means of complex integration, show that

(a). $\int_{-\alpha}^{\alpha} \frac{dx}{1+x^2} = \frac{\pi\sqrt{2}}{2}$

(b). $\int_{0}^{2\pi} \frac{d\theta}{5 - 3\cos\theta} = \frac{\pi}{2}$

(iii). Evaluate $\oint \frac{dz}{(z-a)^n}$; $n = 2, 3, 4, \ldots\ldots\ldots$

Where $z = a$ is inside the simple closed curve c

(iv). Evaluate $\frac{1}{2\pi i}\oint \frac{e^{zt}}{(z^2+1)^2}\,dz$ if $z > 0$ and c is the circle

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Please send the answers to the following address on or before Due date. according to the activity diary

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Model Answer 01 – MPZ 4230

Academic Year 2008

(01). (a). $\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial t}\cdot\dfrac{\partial t}{\partial x}=\dfrac{\partial z}{\partial t}\left[\dfrac{xy-(x+y)y}{x^2y^2}\right]$

$$=\frac{\partial z}{\partial t}\cdot\frac{1}{x^2}$$

$$-x^2\frac{\partial z}{\partial x}=\frac{\partial z}{\partial t} \quad \text{----------- (A)}$$

Then $\dfrac{\partial z}{\partial y}=\dfrac{\partial z}{\partial t}\cdot\dfrac{\partial t}{\partial y}=\dfrac{\partial z}{\partial t}\cdot\left[\dfrac{xy-(x+y)x}{x^2y^2}\right]$

$$=\frac{\partial z}{\partial t}\cdot\frac{-1}{y^2}$$

$$-y^2\frac{\partial z}{\partial y}=\frac{\partial z}{\partial t} \quad \text{--------- (B)}$$

By (A) & (B)

$$-x^2\frac{\partial z}{\partial x}=-y^2\frac{\partial z}{\partial y}$$

$$x^2\frac{\partial z}{\partial x}=y^2\frac{\partial z}{\partial y}$$

(b). Area of the triangle = $A=\dfrac{1}{2}\,ab$

By Chain rule

$$\delta A=\frac{\partial A}{\partial a}\,\delta a+\frac{\partial A}{\partial b}\,\delta b$$

$$=\frac{1}{2}b\;\delta a+\frac{1}{2}\;a\,\delta b$$

$$\delta A\times 100=\frac{1}{2}b.a.\frac{\delta a}{a}\times 100+\frac{1}{2}a.b.\frac{\delta b}{b}\times 100$$

$$=\frac{1}{2}ab\times 0.5+\frac{1}{2}ab\;\;o.5$$

$$\frac{\delta A}{A}\times 100 = \frac{\delta A}{\frac{1}{2}ab}\times 100 = 0.5 + 0.5$$

$$\frac{\delta A}{A}\times 100 = 1$$

maximum possible error in the area of the triangle is 1.

ii. $h^2 = \sqrt{a^2 + b^2}$

$$\delta h = \frac{\partial h}{\partial a}\ \delta a + \frac{\partial h}{\partial a}\ \delta b$$

$$= \frac{1}{2}\frac{2a}{\sqrt{a^2+b^2}}\ \delta a + \frac{1}{2} - \frac{2b}{\sqrt{a^2+b^2}}\delta b$$

$$\delta h \times 100 = \frac{a}{\sqrt{a^2+b^2}}.\ a\ .\ \frac{\delta a}{a}\times 100 + \frac{b}{\sqrt{a^2+b^2}}\ .\ b\ .\ \frac{\delta b}{b}\times 100$$

$$= \frac{a^2}{\sqrt{a^2+b^2}}\ 0.5 + \frac{b^2}{(a^2+b^2)}\ 0.5$$

$$= \frac{0.5}{\sqrt{a^2+b^2}}\ (a^2+b^2)$$

$$= 0.5\ \sqrt{a^2+b^2}\ = 0.5\ h$$

$$\frac{\delta h \times 100}{h} = 0.5$$

Maximum possible error in the hypotenuse is o.5

(02). (a). Let x, y, z be the length of the edges of the box in cm.

Then we have

$$xyz = 32$$

Denote the surface area of the box by S. Then

$$S = xy + 2yz + 2xz$$

Since $z = \frac{32}{xy}$ Then $S = xy + \frac{64}{x} + \frac{64}{y}$

$$\frac{\partial s}{\partial x} = y - \frac{64}{x^2}$$

$$\frac{\partial s}{\partial y} = x - \frac{64}{y^2}$$

Thus the critical points occure when

$$y - \frac{64}{x^2} = 0 \ , \quad x - \frac{64}{y^2} = 0$$

Or equivalently, when

$$x^2 y = 64 \ , \ xy^2 = 64$$

Dividing these equation givens

$$x^3 = 64$$

$$x = 4$$

Thus we have just one critical point, at which $x = y = 4$ and $z = 2$

Now $\Delta = \frac{\partial^2 s}{\partial x^2} \frac{\partial^2 s}{\partial y^2} - \left(\frac{\partial s}{\partial x \ \partial y} \right)^2$

$$= \frac{128}{x^3} \cdot \frac{128}{y^3} - 1 = 0$$

And $\frac{\partial^2 s}{\partial x^2} = \frac{128}{x^3} > 0$

Thus the point is a local minimum and it follows that the dimensions 4cm, 4cm, 2cm gives minimum surface area.

(b). Let the box be x inches deep, y inches long, and z inches wide where x, y, and z are all positive. Then the volume of the box is $v = xyz$ and the total cost of construction is given by

$$c = 1yz + 2(2xy + 2xz) + 5yz$$

$$= 6yz + 4xy + 4xz$$

You wish to minimize $c = 6yz + 4xy + 4xz$ subject to $v = xyz = 96$

The lagrange equations are

$$C_x = \lambda v_x \text{ or } 4y + 4z = \lambda(yz)$$

$$C_y = \lambda v_y \text{ or } 6z + 4x = \lambda(xz)$$

$$C_z = \lambda v_z \text{ or } 6y + 4x = \lambda(xy)$$

And $xyz = 96$. Solve each of the first 3 equations for λ, you get

$$\frac{4y+4z}{yz} = \frac{6z+4x}{xz} = \frac{6y+4x}{xy} = \lambda$$

By multiplying across each equation, you obtain

$$4xyz + 4xz^2 = 6yz^2 + 4xyz$$

$$4xy^2 + 4xyz = 6y^2z + 4xyz$$

$$6xyz + 4x^2y = 6xyz + 4x^2z$$

Which can be further simplified by first subtracting the common xyz terms both. Sides of each equation to get

$$4xz^2 = 6yz^2$$

$$4xy^2 = 6y^2z$$

$$4x^2y = 4x^2z$$

By dividing z^2 from both sides of first equation, y^2 from the second and x^2 from the third, you obtain

$$4x = 6y \text{ and } 4x = 6z \text{ and } 4y = 4z$$

So that $y = \frac{2}{3}x$ and $z = \frac{2}{3}x$, substituting these values into the constraint equation.

$xyz = 96$, you first find that

$$x\left(\frac{2}{3}x\right)\left(\frac{2}{3}x\right) = 96$$

$$\frac{4}{9}x^3 = 96$$

$$x^3 = 216$$

so $x = 6$

and then $y = z = \frac{2}{3}(6) = 4$

Thus the minimal cost occur when the jewel box is 6 inches deep with a square base, 4 inches on a side

Then

$$\frac{\partial^2 c}{\partial x^2} = \frac{276x2}{x^3}$$

$$\frac{\partial^2 c}{\partial y} = \frac{384x2}{y^3}$$

$$\frac{\partial^2 c}{\partial y} = 4$$

$$\Delta = \frac{\partial^2 c}{\partial x^2} \cdot \frac{\partial^2 c}{\partial y^2} - \left(\frac{\partial^2 c}{\partial x \ \partial y} \right)$$

$$\Delta_{(6,4,4)} > 0$$

And $\frac{\partial^2 c}{\partial x^2} > 0$

$\therefore$ This dimensions give the minimum total cost

Method II

(b). Let x, y, z be the length of the edges of the box in cm

Then we have

$$xyz = 96$$

Denote the total cost of the box by C , Then

$$C = yx + 4xy + 4xz + 5yz$$

$$= 6yz + 4xy + 4xz$$

Since $z = \frac{96}{xy}$ then

$$C = \frac{96 \times 6}{x} + 4xy + \frac{96 \times 4}{y}$$

$$\frac{\partial c}{\partial x} = 4y - \frac{576}{x^2}$$

$$\frac{\partial c}{\partial y} = 4x - \frac{384}{y^2}$$

Thus the critical points occur

$$4y - \frac{576}{x^2} = 0 \qquad \& \; 4x - \frac{384}{y^2} = 0$$

$$x^2y = 144 \qquad\qquad xy^2 = 96$$

Solving these equations we can get

$$y^3 = 64$$

$$y = 4$$

Then $x = 6$

Then $z = 4$

We have only one critical point at $x = 6, y = z = 4$

Now $\Delta = \frac{\partial^2 c}{\partial x^2} \cdot \frac{\partial^2 c}{\partial y^2} - \left(\frac{\partial^2 c}{\partial x \, \partial y}\right)^2$

$\Delta_{(6,4,4)} > 0$

And $\frac{\partial^2 c}{\partial x^2} > 0$

$\therefore$ The point is local minimum and it follows that 6 cm, 4 cm, 4 cm gives the minimum total cost.

(03). (1). If curl F = 0, Then F is conservative vectors field.

$$\text{curel } F = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+4z & 2x-3y-z & 4x-y+2z \end{vmatrix}$$

$$= (-1+1)\,\underline{i} + (4-4)\underline{j} + (2-2)\underline{k}$$

$$= 0$$

$\therefore$ F is conservative vector field

(ii). $F = \nabla\phi$

$$(x+2y+4z)\,\underline{i}+(2x\text{-}3y\text{-}z)\,\underline{j}+(4x-y+2z)\underline{k}=\frac{\partial\phi}{\partial x}\underline{i}+\frac{\partial\phi}{\partial y}\underline{j}+\frac{\partial\phi}{\partial z}\underline{k}$$

$$\frac{\partial\phi}{\partial x}=x+2y+4z \quad ----------(1)$$

$$\frac{\partial\phi}{\partial y}=2x-3y-z \quad ---------(2)$$

$$\frac{\partial\phi}{\partial z}=4x-y+2z \quad -----------(3)$$

Let $\frac{\partial\phi}{\partial x}=x+2y+4z$

$$\int\partial\phi=\int(x+2y+4z)\partial x$$

$$\phi=\frac{x^2}{2}+2xy+4xz+f(y,z) \quad ------------(A)$$

Differentiate w.r. to y

$$\frac{\partial\phi}{\partial y}=2x+\frac{\partial f(y,z)}{\partial y}$$

Comparing eq^n (2)

$$\frac{\partial\phi}{\partial y}=2x-3y-z=2x+\frac{\partial f(y,z)}{\partial y}$$

$$-3y-z=\frac{\partial f(y,z)}{\partial y}$$

$$\int(-3y-z)\partial y=\int\partial f(y,z)$$

$$\frac{3y^2}{2}-zy+g(z)=f(y,z)$$

f(y, z) substituting (A)

$$\phi=\frac{x^2}{2}+2xy+4xz-\frac{3y^2}{2}-zy+g(z) \quad ----------- (B)$$

Differentiate w.r. to z

$$\frac{\partial\phi}{\partial z}=4x-y+\frac{\partial g(z)}{\partial z}$$

Comparing eq^n (3)

$$\frac{\partial\phi}{\partial z}=4x-y+2z=4x-y+\frac{\partial g(z)}{\partial z}$$

$$2z = \frac{\partial g(z)}{\partial z}$$

$$\int 2z\,\partial z = \int \partial g(z)$$

$$z^2 + c = g(z)$$

f(z) substituting to (B)

$$\phi = \frac{x^2}{2} + 2xy + 4xz - \frac{3y^2}{2} - zy + z^2 + c$$

(04). (i). Let $z = 8 + 4\sqrt{5}\ i$

$$= \sqrt{8^2 + 4\sqrt{5}^2}\left(\frac{8}{12} + \frac{4\sqrt{5}}{12}i\right)$$

$$z = r\,(\cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi))$$

$$\sqrt{z} = \sqrt{r}\left[\cos\left(\frac{\theta + 2n\pi}{2}\right) + i\sin\left(\frac{\theta + 2n\pi}{2}\right)\right]$$

$$= \sqrt{r}(\cos\left(\frac{\theta}{2} + n\pi\right) + i\sin\left(\frac{\theta}{2} + n\pi\right)$$

Where $r = 12$

and $\theta = \cos^{-1}\left(\frac{2}{3}\right)$ or $\theta = \sin^{-1}\left(\frac{\sqrt{5}}{3}\right)$

(ii). Let $\alpha = x_1 + iy_1$ & $\beta = x_2 + iy_2$

$$|\alpha + \beta|^2 + |\alpha - \beta|^2$$

$$= |(x_1 + iy_1) + (x_2 + iy_2)|^2 + |(x_1 + iy_1) - (x_2 + iy_2)|$$

$$= |(x_1 + x_2) + (y_1 + y_2)i|^2 + |(x_1 - x_2) + i(y_1 - y_2)|^2$$

$$= \left(\sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2}\right)^2 + \left(\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}\right)^2$$

$$= x_1^2 + 2x_1x_2 + x_2^2 + y_1^2 + 2y_1y_2 + y_2^2 + x_1^2 - 2x_1x_2 + x_2^2 + y_1^2 - 2y_1y_2 + y_2^2$$

$$= 2x_1^2 + 2x_2^2 + 2y_1^2 + 2y_2^2$$

$$= 2\left(x_1^2 + y_1^2\right) + 2\left(x_2^2 + y_2^2\right)$$

$$= 2\left(\sqrt{x_1^2 + y_1^2}\right)^2 + 2\left(\sqrt{x_2^2 + y_2^2}\right)^2$$

$$= 2\,|\,\alpha\,|^2 + 2|\,\beta\,|^2$$

(iii). Let $z = x + iy$ then $\bar{z} = x - iy$

Now $\arg z + \arg \bar{z} = \arg \bar{z}\, z$

$$= \arg (x - iy)(x + iy)$$

$$= \arg (x^2 + y^2)$$

$$= \arg a \qquad ; \text{ where } a = x^2 + y^2$$

Clearly a is positive and real. Let $a = r\cos\theta$, $0 = r \sin\theta$ so that $r = a$, $\cos\theta = 1$, $\sin\theta = 0$

So that $r = a$, $\cos\theta = 1$, $\sin\theta = 0$

Therefore the general value of $\theta = 2n\pi$ when n is integer.

(05). (i). Given that $u = x^3 - 3xy^2 - 3x^2 + 3y^2$ ---------------- (A)

Differentiate with respect to x, (A)

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 - 6x \qquad \text{------------------ (1)}$$

d.w.r. to x (1) $\frac{\partial^2 u}{\partial x^2} = 6x - 6$ ----------------- (2)

Then d.w.r. to y (A) $\frac{\partial u}{\partial y} = -6xy + 6y$ ---------------- (3)

d.w.r. to y (3) $\frac{\partial^2 u}{\partial y^2} = -6x + 6$ ----------------- (4)

Then $\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6 - (6x - 6) = 0$

(ii). Let $f(z) = u + iv$ is analytic

for analytic function

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Let $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$

$$\frac{\partial v}{\partial y} = 3x^2 - 3y^2 - 6x$$

$$\int \partial v = \int (3x^2 - 3y^2 - 6x)\partial y$$

$$v = 3x^2y - \frac{3y^3}{3} - 6xy + f(x) \quad \text{-------------} (1)$$

Then let $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$$\frac{\partial v}{\partial x} = 6xy - 6y$$

$$\int \partial v = \int (6xy - 6y)\,\partial x$$

$$V = 6xy - 6xy + f(y) \quad \text{---------------} (2)$$

Comparing (1) & (2)

$V = 3x^2y - 6xy - y^3 + c$ [where $f(x) = c$

$f(y) = -y^3 + c$

$f(z) = (x^3 - 3xy^2 - 3x^2 + 3y^2) + i\,(3x^2y - 6xy - y^3)$

(iii). $f(z) = (x^3 - 3xy^2 - 3x^2 + 3y^2) + i(3x^2y - 6xy - y^3 + c)$

$= (x^3 - iy^3) - (3xy^2 - 3x^2yi) - (3x^2 + 6xyz - 3y^2) + ic$

$= (x^3 + (iy)^3) - (3xy\,(y - xi) - [3x^2 + 6xyi + 3(iy)^2] + c_1$

$= [(x + iy)\,(x^2 - xyi + (iy)^2]\ -3xyi\,(-i^2y\ - xz)\ -3\,(x + iy)^2 + c_1$

$= (x + iy)\,(x^2 - xyi + (iy)^2 + 3xyi\,(x + iy) - 3\,(x + iy)^2 + c_1$

$= (x + iy)\,[x^2 - xyi + (iy)^2 + 3xyi) - 3(x + iy)^2 + c_1$

$= (x + iy)\,(x + iy)^2 - 3\,(x + iy)^2 + c_1$

$= z\,.\,z^2 - 3z^2 + c_1$

$= z^2\,(z - 3) + c_1$

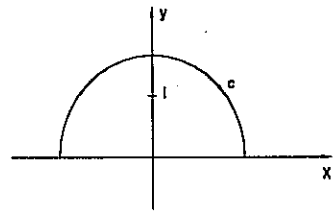
(06).(i). Cauchy's residue theorem

Suppose c is a positively oriented, simple closed contour. If f is analytic on and inside c except for the finite number of singular points $z_1, z_2, z_3, \ldots\ldots z_n$. Then

$$\int_c f(z)\ dz = 2\pi i \sum_{k=1}^{n} \text{Res}_{z=z_k} f(z)$$

(ii). (a).

Let $\int_{-\alpha}^{\alpha} \frac{dz}{1+z^2}$



When $z^2 + 1 = 0$, $\quad z^2 = -1$

$$z = \pm i$$

Here, pole $z = +i$ is inside the circle.

Then Residue $(z - i)$

$\text{Res}_{z=i} f(z) = i$

$$\int_{-\alpha}^{\alpha} \frac{dz}{1+z^2} = \lim_{z \to i} (z-i) \times \frac{1}{(z-i)(z+i)}$$

$$= \lim_{z \to \alpha} \frac{1}{(z+i)}$$

$$= \frac{1}{2i}$$

The applying canchy's residence theorem

$$\int_{-\alpha}^{\alpha} \frac{dz}{1+z^2} = 2\pi i \times \frac{1}{2i}$$

$$= \pi$$

(ii). (b). $\int_0^{2\pi} \frac{d\theta}{5 - 3\cos\theta}$

Let $z = e^{i\theta}$

$$\cos\theta = \frac{(e^{i\theta} + e^{-i\theta})}{2}$$

$$=\frac{1}{2}\left(z+\frac{1}{z}\right)$$

$$\frac{dz}{d\theta}=ire^{i\theta}=iz$$

Then

$$\int_0^{2\pi}\frac{d\theta}{5-\cos\theta}=\int_0^{2\pi}\frac{dz/iz}{5-\frac{3}{2}\left(z+\frac{1}{z}\right)}$$

$$=\frac{1}{i}\int_0^{2\pi}\frac{dz}{5z-\frac{3z^2}{2}-\frac{3}{2}}$$

$$=-\frac{2}{3i}\int_0^{2\pi}\frac{dz}{(z-3)\left(z-\frac{1}{3}\right)}$$

Consider pole $z=\frac{1}{3}$

Then residue $Res_{r\to 1/3}=\frac{1}{3}$

$$\int_0^{2\pi}\frac{d\theta}{5-3\cos\theta}=\lim_{z\to 1/3}\left(z-\frac{1}{3}\right)\times\frac{-2/3}{\left(z-1/3\right)(z-3)}=\frac{1}{4}$$

Consider pole $z=3$

Then residue $Res_{z\to 3}=3$

$$\int_0^{2\pi}\frac{d\theta}{5-3\cos\theta}=\lim_{z\to 3}(z-3)\times\frac{-2/3}{\left(z-1/3\right)(z-3)}=-\frac{1}{4}$$

Then applying caushy's residue theorem

$$\int_0^{2\pi}\frac{d\theta}{5-3\cos\theta}=\frac{1}{i}\times 2\pi i\left\{\frac{1}{4}-\frac{1}{4}\right\}$$

$$=\frac{\pi}{2}-\frac{\pi}{2}$$

$$=0$$

(iii). $\oint \frac{1}{(z-a)^n} dz$; $n = 2, 3, 4, \ldots\ldots\ldots$

(where z = a is inside the simple closed curve c)

A singularity occur at z = a

Let $z = re^{i\theta}$

$dz = ire^{i\theta}$

Here c enclosed G (z = a) we consider the case when $n > 1$.

$$\oint f(z)dz = \oint \frac{1}{(z-a)^n} dz = \int_0^{2\pi} \frac{1}{(re^{i\theta}-a)^n} ire^{i\theta} \; d\theta$$

$$= \int_0^{2\pi} (ire^{i\theta})(re^{i\theta}-a)^{-n} d\theta$$

$$= ir\left[\frac{(re^{i\theta}-a)^{-(n-1)}}{ir(n-1)}\right]_0^{2\pi}$$

$$= \frac{1}{-(n-1)}\left[\{r(\cos\theta + i\sin\theta) - a\}^{-(n-1)}\right]_0^{2\pi}$$

$$= \frac{1}{-(n-1)}\left[\{r(\cos 2\pi - i\sin 2\pi) - a\}^{-n-1} - \{r(\cos\theta + i\sin\theta - a)^{-(n-1)}\}\right]$$

$$= \frac{-1}{n-1}\{(r-a)^{-(n-1)} - (r-a)^{-(n-1)}\}$$

$$= \frac{-1}{(n-1)}\left\{\frac{1}{(r-a)^{n-1}} - \frac{1}{(r-a)^{n-1}}\right\}$$

So $\oint \frac{1}{(z-a)^n} dz = 0$ for all positive value of n other than n = 1.

(ii). (d).