MPZ 4230 - Assignment No. 03 – Academic Year 2008

1). (a). Define a Fourier series of a function f(x) for $-L < x < L$ with period 2L.

(b).

$$\text{Let} \quad f(t) = \begin{cases} -1 & ; -\pi < t < -\dfrac{\pi}{2} \\ 1 & ; -\dfrac{\pi}{2} < t < \dfrac{\pi}{2} \\ -1 & ; \dfrac{\pi}{2} < t < \pi \end{cases}$$

(i). Sketch the graph of f(t)
(ii). Classify the function f(t) as even , odd or neither even nor odd
(iii). Obtain the Fourier Series expansion of f(t).
(iv). Show that $1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + ... = \dfrac{\pi}{4}$

2). (a). Determine the given functions are odd, even or neither even nor odd
(i). $f(t) = t$
(ii). $f(t) = t^2$
(iii). $f(t) = t^3$
(iv). $f(t) = \sin t$
(v). $f(t) = t^2 + t^3$

(b). Obtain a Fourier Series for the function $f(x) = |\cos x|$.
Where $-\pi < x < \pi$

3). Motion of a body with coordinate y(t) is periodic with period 2π and satisfied the following differential equation.

$$\frac{d^2 y}{d t^2} + 0.02\frac{dy}{dt} + 25y = f(t)$$

Where the force f(t) is 2π period and

$$f(t) = \begin{cases} t + \dfrac{\pi}{4} & -\pi < t < 0 \\ -t + \dfrac{\pi}{4} & 0 < t < \pi \end{cases}$$

4). Use the RLC series circuit model to determine the capacitor voltage and current in a circuit with $L = 10$ H
$R = 1000\,\Omega$, $C = 10\,\mu$ F, and $E = 3v$, assuming that the circuit is initially at rest.

(05). Find the radius of convergence of the following power series.

(i). $\sum_{n=1}^{\alpha} n^n x^n$

(ii). $\sum_{n=0}^{\alpha} \frac{x^n}{n!}$

(iii). $\sum_{n=0}^{\alpha} \frac{(-1)^{2n+1} x^{2n+1}}{(2n+1)\ !}$

(iv). $\sum_{n=0}^{\alpha} \frac{(x-1)^n}{2^n (2n+1)}$

(v). $\sum_{n=1}^{\alpha} \frac{(x-1)^n}{n(n+1)}$

MPZ 4230 – Model Answer 03
Academic Year 2008

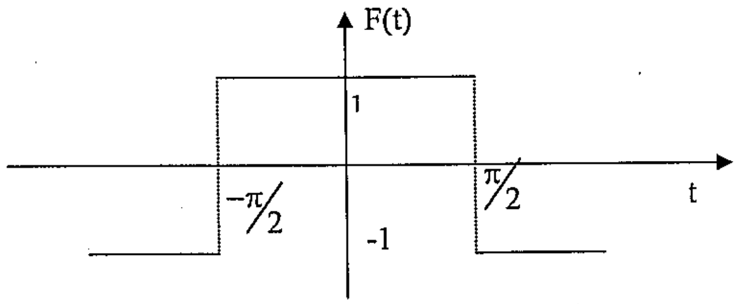
(01). (a) $f(x) = \frac{a_0}{2} + \sum a_n \cos nx + \sum b_n \sin nx$

Where $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\ \cos\ \frac{n\pi x}{L} dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x).\ \sin \frac{n\pi x}{L} dx$$

(b).



(c). f(t) = f(-t)

∴ f(t) is a even function

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t).\ dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{-\pi/2} -1\ dt + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 1\,dt + \frac{1}{\pi}\int_{\pi/2}^{\pi} -1\ dt$$

$$= \frac{-1}{\pi}[t]_{-\pi}^{-\pi/2} + \frac{1}{\pi}\int_{-\pi/2} [t]_{-\pi/2}^{\pi/2} \pm \frac{1}{\pi}[t]_{\pi/2}^{\pi}$$

$$= -\frac{1}{\pi}\left[\frac{\pi}{2} - \pi\right] + \frac{1}{\pi}\left[\frac{\pi}{2} + \frac{\pi}{2}\right] - \frac{1}{\pi}\left[x - \frac{\pi}{2}\right]$$

$$= 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\ \cos nt\ dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2} -\cos\ nt\ dt + \int_{-\pi/2}^{\pi/2} \cos\ nt\ dt + \int_{\pi/2}^{\pi} -\cos\ nt\ dt\right]$$

$$= -\frac{1}{\pi}\left[\frac{\sin nt}{n}\right]_{-\pi}^{-\pi/2} + \frac{1}{\pi}\left[\frac{\sin nt}{n}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \frac{1}{\pi}\left[\frac{\sin nt}{n}\right]_{\pi/2}^{\pi}$$

$$= \frac{-1}{n\pi}\left[\sin\left(-\frac{n\pi}{2}\right) - \sin(-n\pi)\right] + \frac{1}{n\pi}\left[\sin\frac{n\pi}{2} - \sin\left(-\frac{n\pi}{2}\right)\right] - \frac{1}{n\pi}\left[\sin n\pi - \sin\frac{n\pi}{2}\right]$$

$$= -\frac{2}{n\pi}\left[-\sin\frac{n\pi}{2} + \sin n\pi\right] + \frac{1}{n\pi}\left[\sin\frac{n\pi}{2} + \sin\frac{n\pi}{2}\right] - \frac{1}{n\pi}\left[\sin n\pi - \sin\frac{n\pi}{2}\right]$$

$$= \frac{1}{n\pi}\left[-2\sin n\pi + 4\sin\frac{n\pi}{2}\right]$$

$$= 4n\pi\sin\frac{n\pi}{2}$$

$a_1 = \frac{4}{\pi}$ $\quad a_2 = 0$ $\quad a_3 = \frac{-4}{3\pi}$ $\quad a_4 = 0$ $\quad a_5 = \frac{-4}{5\pi}$ $\quad a_6 = 0$

$$f(t) = \frac{4}{\pi}\left[\cos t - \frac{\cos 3t}{3} + \frac{\cos 5t}{5} - \frac{\cos 7t}{7} + \dots\dots\dots\dots\right]$$

(iv). when $t = 0$ $\quad f(t) = 1$

$$1 = \frac{4}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{a} + \dots\dots\right]$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \dots\dots$$

(02). (a). (i). $f(t) = t$

$f(-t) = -t$

$-f(-t) = t$

$f(t) = -f(-t)$

$\therefore$ f(t) is a odd function

(ii). $f(t) = t^2$

$f(-t) = (-t)^2 = t^2$

$f(t) = f(-t)$

$\therefore$ f(t) is a even function

(iii). $f(t) = t^3$

$f(-t) = (-t)^2 = -t^3$

$-f(-t) = t^3$

$f(t) = -f(-t)$

$\therefore$ f(t) is a odd function

(iv). $f(t) = \sin t$

$f(t) = \sin(-t) = -\sin t$

$-f(-t) = \sin t$

$f(t) = -f(-t)$

$\therefore$ f(t) is a odd function

(v). $f(t) = t^2 + t^3$

$f(-t) = (-t)^2 + (-t)^3 = t^2 - t^3$

$-f(-t) = -t^3 + t^3$

$f(t) \neq f(-t) \neq -f(-t)$

$\therefore$ f (t) is not a even nor odd function

(3). Replace f (t) by its Fourier series

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt\infty$$

$$f(t) = \begin{cases} t + \dfrac{\pi}{2} & -\pi < t < 0 \\ -t + \dfrac{\pi}{2} & 0 < t < \pi \end{cases}$$

$$f(-t)=\begin{cases}-t+\dfrac{\pi}{2} & -\pi<t<0\\ t+\dfrac{\pi}{2} & 0<t<\pi\end{cases}$$

f(t) = f(-t)

$\therefore$ f(t) is a even function

$\therefore b_n = 0$

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t).dt$$

$$=\frac{1}{2\pi}\left[\frac{t^2}{2}+\frac{t\pi}{2}\right]_{-\pi}^{0}+\frac{1}{2\pi}\left[-\frac{t^2}{2}+\frac{t\pi}{2}\right]_{0}^{\pi}$$

$$=-\frac{1}{2\pi}\left[\frac{tx^2}{2}-\frac{\pi^2}{2}\right]+\frac{1}{2\pi}\left[-\frac{\pi^2}{2}+\frac{\pi^2}{2}\right]$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt$$

$$=\frac{1}{\pi}\int_{-\pi}^{0}\left(t+\frac{\pi}{2}\right)\cos ntdt+\frac{1}{\pi}\int_{0}^{\pi}\left(-t+\frac{\pi}{2}\right)\cos nt\,dt$$

$$=\frac{1}{2}\underbrace{\left[\frac{\sin nt}{n}\right]_{-\pi}^{0}}_{0}-\frac{1}{\pi}\int_{-\pi}^{0} t\cos nt\,dt+\frac{1}{2}\underbrace{\left[\frac{\sin nt}{n}\right]_{0}^{\pi}}_{0}+\frac{1}{\pi}\int t\cos nt\,dt$$

$$=\left[\frac{t\sin nt}{n\pi}\right]_{-\pi}^{0}+\left[\frac{\cos nt}{\pi n^2}\right]_{-\pi}^{0}-\left[\frac{t\sin nt}{n\pi}\right]_{0}^{\pi}-\left[\frac{\cos nt}{\pi n^2}\right]_{0}^{\pi}$$

$$=\frac{1}{\pi n^2}-\frac{\cos n\pi}{\pi n^2}+\frac{1}{\pi n^2}-\frac{\cos n\pi}{\pi n^2}$$

$$=\begin{cases}0 & ;\text{n is even}\\ \dfrac{4}{\pi n^2} & ;\text{n is odd}\end{cases}$$

$$f(t) = \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \dots\dots\right) \quad \text{-----------}(1)$$

Then differentiate unknown Fourier series

$$y(t) = A_0 + \sum_{n=1}^{\infty}(A_n \cos nt + B_n \sin nt)$$

then

$$y'(t) = \sum_{n=1}^{\infty}\left[n\, B_n \cos(nt) + (-nA_n)\sin nt\right]$$

$$y''(t) = \sum_{n=1}^{\alpha}\left[\left(-n^2 A_n \cos nt + \left(-n^2 B_n\right)\sin(nt)\right)\right]$$

Then

$$y'' + 0.02y'(t) + 25y = 25A_0 + \sum_{n=1}^{\infty}\left(-n^2 A_n + 0.02nB_n + 25A_n\right)\cos nt$$

$$+ \sum_{n=1}^{\infty}\left(-n^2 B_n - 0.02nA_n + 25B_n\right)\sin nt \quad \text{--------} (B)$$

Comparing (1) and (2)

$25\ A_0 = 0$
$\therefore\ A_0 = 0$

For the sine and cosine terms with even n we have

$(25 - n^2)\ A_n + 0.02n\ B_n = 4/\pi n^2$ --------------- (3)
$-0.02\ n\ A_n + (25 - n^2)\ B_n = 0$ --------------- (4)

$A_n = B_n = 0$
For the sine and cosine terms with odd n, we have
$(25 - n^2)\ A_n + 0.02n\ B_n = 0$ ----------------- (5)
$0.02\ n\ A_n + (25 - n^2)\ B_n = 0$ ----------------- (6)

Solving (5) & (6)

$$A_n = \frac{4\left(25 - n^2\right)}{n^2\pi\left[\left(25 - n^2\right)^2 + (0.02n)^2\right]}$$

$$B_n = \frac{0.08}{n\pi\left[\left(25 - n^2\right)^2 + (0.02n)^2\right]}$$

Then we have obtain solution in the form

$$y(t) = \sum_{n=1,3,\ldots\ldots} (A_n \cos nt + B_n \sin nt)$$

Where A_n and B_n are defined by the above formula depending on n

n	A_n	B_n
1	0.0531	0.0000
3	0.0088	0.0000
5	0.0000	0.5093
7	-0.0011	0.0000
9	-0.0003	0.0000

(04). Application of Kirchhoff's voltage low to series circuit yield

$$V_L + V_R + V_C = E \qquad \text{------------------ (1)}$$

$$L\frac{di}{dt} + iR + \frac{Q}{C} = E$$

By differentiating

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}\frac{dQ}{dt} = 0$$

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = 0$$

$$\frac{d^1i}{dt^2} + \frac{1000}{10}\frac{di}{dt} + \frac{1}{10\times10\times10^{-6}} = 0$$

General solution is $i = Ae^{\lambda t}$

$$\frac{di}{dt} = \lambda Ae^{\lambda t}$$

$$\frac{d^2i}{dt^2} = \lambda^2 Ae^{\lambda t}$$

$$\lambda^2 Ae^{\lambda t} + 100\lambda Ae^{\lambda t}\ 10^4\ Ae^{\lambda t} = 0$$

$$Ae^{\lambda t}[\lambda^2 + 10\lambda + 10^4] = 0$$

$$\lambda = \frac{-100 \pm \sqrt{10^4 - 4\times10^4}}{2}$$

$$= \frac{-100 \pm 100\sqrt{-3}}{2}$$

$$= -50 \pm 50\sqrt{3}\ i$$

$$i = e^{-50t}(A\cos 50\sqrt{3}\,t + B\sin 50\sqrt{3}\,t)$$

When $t = 0$ $i = 0$

$$0 = 1\ (A\cos 0 + B\sin 0)$$

$$A = 0$$

$$\therefore\ i = e^{-50t} B\sin 50\sqrt{3}\ t$$

$$\frac{di}{dt} = B\left[-50e^{-50t}\sin\ 50\sqrt{3}\,t + e^{-50t}50\sqrt{3}\cos 50\sqrt{3}\,t\right]$$

$t = 0^+ \quad \frac{di}{dt} = \frac{V_0}{L}$

$$\frac{3}{10} = B.\ 50\sqrt{3}$$

$$B = \frac{3}{500\sqrt{3}}$$

$$= \frac{\sqrt{3}}{500}$$

$$\therefore\ i(t) = \frac{\sqrt{3}}{500}\sin 50\sqrt{3}\,t\,e^{-50t}$$

$$\frac{di}{dt} = \frac{\sqrt{3}}{500}e^{-50t}\left[-50\sin\ 50\sqrt{3}\ t\ + 50\sqrt{3}\ \cos\ 50\sqrt{3}\ t\right]$$

By (1)

$$V_C = E - V_L - V_R$$

$$= 3 - L\frac{di}{dt} - iR$$

$$= 3 - 10\frac{\sqrt{3}}{500}e^{-50t}\left[-50\sin\ 50\sqrt{3}\ t\ + 50\sqrt{3}\ \cos\ 50\sqrt{3}\ t\right] - 1000 \times \frac{\sqrt{3}}{500}\sin\ 50\sqrt{3}\ t\,e^{-50t}$$

$$= 3 - \left[3\cos 50\sqrt{3}\ e^{-50t} - \sqrt{3}\sin 50\sqrt{3}\ te^{-5t}\right]$$

$$= 3 - e^{-50t}\left(3\ \cos\ 50\sqrt{3}\ t + \sqrt{3}\ \sin\ 50\sqrt{3}\ t\right)$$

(05). (i). by using ratio test

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{(n+1)\ \text{term}}{n\ \text{term}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^{n+1}\ x^{n+1}}{n^n\ \ x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)^n(n+1)}{n^n}\right|\ |x|$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n(n+1)\ |x|$$

$$= \infty\ |x| \le 1$$

Radius of convergence is $R = 0$

Series converge when $x = 0$

(ii). by using ratio rest

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{(n+1)\ \text{term}}{n\ \text{term}}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right|$$

$$= \lim_{n\to\infty}\frac{1}{n+1}|\,x\,|$$

$$= 0\,|\,x\,| \le 1$$

Radius of convergence is $R = \infty$

Series convergence for all value of x

(iii). Using ratio test

$$\frac{1}{R} = \operatorname{Lim}_{n\to\infty}\left|\frac{(n+1)\ \text{term}}{n\ \text{term}}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+3}}{(2n+3)!}\cdot\frac{(2n+1)!}{x^{2n+1}}\right|$$

$$= \lim_{n\to\infty}\frac{x^2}{(2n+3)(2n+2)}$$

$$= \lim_{n\to\infty}\left|\frac{x^2}{4n^2+10n+6}\right|$$

$$= 0\left|x^2\right| \le 1$$

Radius of convergence is $R = \infty$

Series convergence for all value of x

(iv).. by using ratio rest

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{(n+1)\ \text{term}}{n\ \text{term}}\right| = \lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{2^{n+1}(2n+3)}\ \frac{2^n(2n+1)}{(x-1)^n}\right|$$

$$= \lim_{n\to\infty}\frac{2n+1}{2(2n+3)}|\,x-1\,|$$

$$= \frac{|x-1|}{2} \le 1$$

Radius of convergence is $R = 2$

Open interval of convergence $-2 \le x-1 \le 2$

$$-1 \le x \le 3$$

(v). by using ratio test

$$\operatorname{Lim}_{n\to\infty}\left|\frac{(n+1)\ \text{term}}{n\ \text{term}}\right| = \lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{(n+1)(n+2)}\cdot\frac{n(n+1)}{(x-1)^n}\right|$$

$$= \lim_{n\to\infty}\left|(x-1)\cdot\frac{n}{n+1}\right|$$

$$= |\,x-1\,| \le 1$$

Radius of convergence is $R = 1$

Open interval of convergence is $0 \le x \le 2$