MPZ 4230 - Assignment No. 04 – Academic Year 2008

(01). (a). Determine whether or not the given subset S is a subspace of the real vector space V

(i). V= $\mathbb{R}^3$ and $S = \{(x, y, z): x + z = y\}$

(ii). V = P_4 (real polynomial of degree < 5) and S is the set of all polynomials of degree = 2

(iii). V = P_4 and S = the set of all even degree polynomials of P_4

(b). Write down a basis for the real vector space P_3. Find the dimension of the subspace generated by the vector x, $x - 1$, $x^2 + 1$.

(02). State whether the following are true or false. Justify your answer

(a). Let $S = \{u, v, w\}$ be a set of three vectors in $\mathbb{R}^3$. If none of the vectors in S is a multiple of another vector, then S is linear independent

(b). Let $A = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & -1 & -2 & 0 \end{bmatrix}$

Then the columns of A are linearly independent.

(03). T is a linear transformation of $\mathbb{R}^3$ into $\mathbb{R}^3$, such that

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) \quad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \left(\begin{bmatrix}1\\2\\1\end{bmatrix}\right) \quad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \left(\begin{bmatrix}2\\3\\1\end{bmatrix}\right)$$

Find T and Kernel of T

(04). (a). The vectors $u_1 = (1, 0, 0, 1)$, $u_2 = (-1, 0, 2, 1)$, $u_3 = (2, 3, 2, -2)$, $u_4 = (-1, 2, -1, 1)$ and linearly independent $\mathbb{R}^4$. Consider $\mathbb{R}^4$ as a real inner product space with the Euclidean inner product and apply the Gram-Schmidt process this basis.

(b). Using above vectors verify the theorem [theorem; if $\underline{u}$ and $\underline{v}$ are orthogonal vectors in an inner product space, then $\| \underline{u} + \underline{v} \|^2 = \| \underline{u} \|^2 + \| \underline{v} \|^2$

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(02). State whether the following are true or false. Justify your answer

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MPZ 4230 – Model Answer 04
Academic Year 2008

(01). (a). (i). $v = \mathbb{R}^3$ & $S = \{ (x, y, z) : x + z = y \}$
We want to prove
If k is any scalar and $\underline{u}$, $\underline{w}$ $\in S \Rightarrow \alpha\underline{u} + \beta\underline{w} \in S$ *for* $\alpha, \beta \in k$
$u = (x_1, y_1, z_1)$; where $x_1 + z_1 = y_1$
$w = (x_2, y_2, z_2)$; where $x_2 + z_2 = y_2$
$\alpha\underline{u} + \beta\underline{w} = \alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2)$
$= ((\alpha x_1 + \beta x_2), (\alpha y_1 + \beta y_2), (\alpha z_1 + \beta z_2))$

$$x_3 = \alpha x_1 + \beta x_2$$

$= (x_3, y_3, z_3)$ *where* $y_3 = \alpha y_1 + \beta y_2$

$$z_3 = \alpha z_1 + \beta z_2$$

Then find $x_3 + z_3 = \alpha x_1 + \beta x_2 + \alpha z_1 + \beta z_2$
$= \alpha(x_1 + z_1) + \beta(x_2 + z_2)$
$= \alpha y_1 + \beta y_2$
$= y_3$

$\therefore \alpha\underline{u} + \beta\underline{w} \in S$ $[\because x_3 + z_3 = y_3]$
$\therefore$ S is a subspace of $v = \mathbb{R}^3$

(ii). S is set of polynomial degree = 2
Let $u_1, u_2 \in S$ and $a_1 = 1$ & $a_2 = -1$
$u_1 = a_1 x^2 + b_1 x + c_1$
$u_2 = a_1 x^2 + b_2 x + c_2$
$u_1 + (-1) u_2 = (b_1 - b_2) x + (c_1 - c_2) = bx + c \notin S$
$\therefore$ set of polynomial of degree = 2 is not a sub space of p_4

(iii). S is set of all even degree polynomial
Let $u_1, u_2 \in S$ and $a_1 = 1$ & $a_2 = -1$
$u_1 = a_1 x^2 + b_1 x + c_1$
$u_2 = a_2 x^2 + b_2 x + c_2$
$u_1 + (-1) u_2 = (b_1 - b_2) x + (c_1 - c_2) \notin S$
$\therefore$ set of polynomial of even degrees is not a subspace of P_4

b) Basis - $a_1x^3 + b_1x^2 + c_1x + d_1$, $b_2x^2 + c_2x + d_2$, $c_3x + d_3$, d_4
Dimension of sub space= 3

(02). (a). False

Let u = (1,0,0) v = (0,1,0) & w = (1, 1, 0)

Then (1, 0, 0) $\neq$ α (0, 1, 0)

(0, 1, 0) $\neq$ β (1, 1, 0) [where α, β, $\gamma \in \mathbb{R}$]

(1, 0, 0) $\neq$ γ(1, 1, 0)

$\therefore$ non of the vectors of u, v, w in S multiple of other

but $1\underline{u}+1\underline{v}-1w=0$

$\therefore$ u, v, w are dependent

(b). True

a (1, 1, 2, 1) + b(-1, 0, 1, -1) + c(2, 1, 1, -2) + d(0, 1, 0, 0) = 0

$$\begin{pmatrix} 1 & -1 & 2 & 0 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$R_4 \to R_4 - R_1$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - 2R_2$

$$\begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 3 & -3 & 0 \\ 0 & 2 & -4 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$R_3 \to R_3 - 3R_2$

$R_4 \to R_4 - 2R_2$

$$\begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & -2 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving,

by 3rd row d=0

4^{th} row c=0

2^{nd} row b=0

First row a=0

a, b, c, d all are zero

$\therefore$ given vector are linearly independent

(3). We first express $x = (x_1, x_2, x_3)$ as a linear combination of
$\underline{v}_1 = (1, 0, 0)$, $\underline{v}_2 = (0, 1, 0)$ & $\underline{v}_3 = (0, 0, 1)$
$(x_1, x_2, x_3) = k_1 (1, 0, 0) + k_2 (0, 1, 0) + k_3 (0, 0, 1)$

$x_1 = k_1$
$x_2 = k_2$
$x_3 = k_3$

$(x_1, x_2, x_3) = x_1 (1, 0, 0) + x_2 (0, 1, 0) + x_3(0, 0, 1)$

Taking linear transformation

$$\begin{aligned} T(x_y\ x_z, x_3) &= x_1 T(1, 0, 0) + x_2 T(0, 1, 0) + x_3 T(0, 0, 1) \\ &= x_1 (0, 1, 1) + x_2 (1, 2, 1) + x_3 (2, 3, 1) \\ &= (x_2 + 2x_3, x_1 + 2x_2 + 3x_3, x_1 + x_2 + x_3) \end{aligned}$$

Ker $T = \{(x_1, x_2, x_3) \in \mathbb{R}^3 ; T(x_1, x_2, x_3) = 0\}$

$x_2 + 2x_3 = 0$ -------- (1)
$x_1 + x_2 + x_3 = 0$ -------- (2)
$x_1 + 2x_2 + 3x_3 = 0$ -------- (3)

(2) × 3 – (3) $\quad 2x_1 + x_2 = 0$
$x_2 = -2x_1$
(1) $\quad x_2 = -2x_3$

Ker $T = \{x_2 (-1, 2, -1) ; x_2 \in \mathbb{R}\}$

(4). Step 1

$$\underline{v}_1 = \frac{\underline{u}_1}{\|\underline{u}_1\|} = \frac{(1,0,0,1)}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}},\ 0,\ 0, \frac{1}{\sqrt{2}}\right)$$

Step 2

$$\underline{v}_2 = \frac{\underline{u}_2 - \text{proj}\ \underline{w}_1 \underline{u}_2}{\|\underline{u}_2 - \text{proj}\ \underline{w}_1 \underline{u}_2\|}$$

$$\begin{aligned} \underline{u}_2 - \text{proj}\ \underline{w}_1\ \underline{u}_2 &= \underline{u}_2 - (\underline{u}_2 \cdot \underline{v}_1)\ \underline{v}_1 \\ &= (-1, 0, 2, 1) - 0\underline{v}_1 \\ &= (-1, 0, 2, 1) \end{aligned}$$

$$\underline{v}_2 = \frac{-1,\ 0, 2,\ 1}{\sqrt{6}} = \left(\frac{-1}{\sqrt{6}}, 0, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

Step 3

$$\underline{v}_3 = \frac{\underline{u}_3 - \text{proj}\ \underline{w}_2 \underline{u}_3}{\|\underline{u}_3 - \text{proj}\ \underline{w}_2 \underline{u}_3\|}$$

$$\begin{aligned} \underline{u}_3 - \text{proj}\ \underline{w}_2\ \underline{u}_2 &= \underline{u}_3 - (\underline{u}_3 \cdot \underline{v}_1)\ \underline{v}_1 - (\underline{u}_3 \cdot \underline{v}_2)\ \underline{v}_2 \\ &= (-2, 3, 2, -2)\ - 0\underline{v}_1,\ - 0\underline{v}_2 \end{aligned}$$

$$\underline{v}_3 = \frac{(2,\ 3,\ 2,\ -2)}{\sqrt{21}} = \left(\frac{2}{\sqrt{21}},\ \frac{3}{\sqrt{21}},\ \frac{2}{\sqrt{21}},\ \frac{-2}{\sqrt{21}}\right)$$

Step 4

$$\underline{v}_4 = \frac{\underline{u}_4 - \text{proj } \underline{w}_3\underline{u}_4}{\left\|\underline{u}_4 - \text{proj } \underline{w}_3\underline{u}_4\right\|}$$

$$\underline{u}_4 - \text{proj } \underline{w}_3\underline{u}_4 = \underline{u}_4 - (\underline{u}_4 \cdot \underline{v}_1)\ \underline{v}_1 - (\underline{u}_4 \cdot \underline{v}_2)\ \underline{v}_2 - (\underline{u}_4 \cdot \underline{v}_3)\ \underline{v}_3$$

$$= (-1, 2, -1, 2) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}},\ 0,\ 0,\ \frac{1}{\sqrt{2}}\right) - \frac{2}{\sqrt{6}}\left(-\frac{1}{\sqrt{6}},\ 0,\ \frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$$

$$-\frac{7}{\sqrt{21}}\left(\frac{2}{\sqrt{21}},\ \frac{3}{\sqrt{21}},\ \frac{2}{\sqrt{21}}, \frac{-2}{\sqrt{21}}\right)$$

$$= \left(-\frac{1}{6},\ 3, \frac{1}{3}, \frac{7}{6}\right)$$

Consider $\underline{v}_1$ and $\underline{v}_2$ are orthogonal vectors

$$\underline{v}_1 + \underline{v}_2 = \left(\frac{\sqrt{3}-1}{\sqrt{6}}, \frac{0}{-}, \frac{2}{\sqrt{6}}, \frac{\sqrt{3}+1}{\sqrt{6}}\right)$$

$$\left\|\underline{v}_1 + \underline{v}_2\right\| = \sqrt{\left(\frac{3-2\sqrt{3}+1+4+3-2\sqrt{3}+1}{6}\right)}$$

$$\left\|\underline{v}_1 + \underline{v}_2\right\| = 2$$

$$\left\|\underline{v}_1\right\| = 1$$

$$\left\|\underline{v}_2\right\| = 1$$

$$\therefore \left\|\underline{v}_1 + \underline{v}_2\right\|^2 = \left\|\underline{v}_1\right\|^2 + \left\|\underline{v}_2\right\|^2$$

(05). $A - \lambda I = \begin{bmatrix} 2-\lambda & -4 & 0 \\ -4 & -\lambda & 4 \\ 0 & 4 & -2-\lambda \end{bmatrix}$

$= (2 - \lambda)\ [-\lambda\ (-2 - \lambda) - 16]\ + 4\ [-4\ (-2 - \lambda)] = 0$
$= (2 - \lambda)\ [2\lambda + \lambda^2 - 16] + 16(2 + \lambda) = 0$
$= 4\lambda + 2\lambda^2 - 32 - 2\lambda^2 + \lambda^3 + 16\lambda + 32 + 16\lambda$
$= -\lambda^3 + 36\lambda$
$= \lambda\ (\lambda - 6)\ (\lambda + 6) = 0$

$\lambda = 0$ or $\lambda = -6$ or $\lambda = 6$

b) When $\lambda = 0$

$A - \lambda I = 0$

$$\begin{pmatrix} 2 & -4 & 0 \\ -4 & 0 & 4 \\ 0 & 4 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$2a - 4b = 0$
$-4a + 4c = 0$
$4b - 2c = 0$

Let $b = t$
Then $a = 2t$
$c = 2t$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} t = v_1 t$$

v_1 is a eigen vector only handling legers

When $\lambda = 6$

$(A - \lambda I)\, x = 0$

$$\begin{pmatrix} -4 & -4 & 0 \\ -4 & -6 & 4 \\ 0 & 4 & 8 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$-4a - 4b = 0$
$-4a - 6b + 4c = 0$
$4b - 8c = 0$

let $c = t$
then $b = 2t$
$a = -2t$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} t = v_2 t$$

v_2 is a eigen vector

When $\lambda = -6$

(A - λI) x = 0

$$\begin{pmatrix} 8 & -4 & 0 \\ -4 & 6 & 9 \\ 0 & 4 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$8a - 4b = 0$$
$$-4a + 6b + 4c = 0$$
$$4b + 4c = 0$$

Let a = t

Then b = 2t

c=-2t

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} t = v_3 t$$

V_3 is a eigen vector

c)

$$\text{let } p = \begin{pmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{pmatrix}$$

$$|p| = 2(-6) - 2(6) + 1(-3) = -27$$

$$p^{-1} = \frac{1}{-27}\begin{pmatrix} -6 & -3 & -6 \\ 6 & -6 & -3 \\ -3 & -6 & 6 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 2 \\ -2 & 2 & 1 \\ 1 & 2 & -2 \end{pmatrix}$$

$$p^{-1} AP = \begin{pmatrix} 2 & 1 & 2 \\ -2 & 2 & 1 \\ 1 & 2 & -2 \end{pmatrix}\begin{pmatrix} 2 & -4 & 0 \\ -4 & 0 & 4 \\ 0 & 4 & -2 \end{pmatrix}\begin{pmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & 2 \\ -2 & 2 & 1 \\ 1 & 2 & -2 \end{pmatrix}\begin{pmatrix} 0 & -12 & -6 \\ 0 & 12 & -12 \\ 0 & 6 & 12 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

(d). let S = P^{-1} AP

$SS^{-1} = (p^{-1}AP)S^{-1}$
$I = p^{-1}\ APS^{-1}$
$PI = APS^{-1}$
$A^{-1}P^{-1} = PS^{-1}$
$P^{-1}AP\ = S^{-1}$
$\therefore\ S^{-1} = P^{-1}A^{-1}P$

(e). $Q(x) = x^2_1 - 8x_1\, x_2 + 8x_2\, x_3 + x_3^2$

$$\therefore \text{ Given Quadratic form is } Q(x) = (x_1\ x_2\ x_3) \begin{pmatrix} 2 & -4 & 0 \\ -4 & 0 & 4 \\ 0 & 4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\text{The new quadratic form is } Q(y) = (y_1\ y_2\ y_3) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$
$$= 6y^2_2 - 6y_3^2$$

(d). $x = py$

$Q(x)\ x^T A\ x = x^T\ \underbrace{pp^{-1}App^{-1}}\ \ x$

$= \ x^T\, p\, \lambda\, p^{-1}\, x$
$= \ \underbrace{x^T p}\ \lambda\, p^T\, x$

$Q(y)\ \ = y^T\, \lambda\, y$

Relation $\ \ y = p^T\, x$
$y^T = (p^T\, x)^T$
$x^T\, p$