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THE OPEN UNIVERSITY OF SRI LANKA
BACHELOR OF TECHNOLOGY (level 05)
ECX 5241
DISTRIBUTED PARAMETER SYSTEMS
FINAL EXAMINATION 2009

DATE : 14th March 2010 **TIME : 09.30 – 12.30 hours**

Select **ONE** question each from Sections **A** and **B** and answer **all questions** in Section C.

SECTION A:
Select **ONE** question

A1.
The temperature distribution between the region of two concentric spherical surfaces of radii r_1 and r_2 ($r_1 > r_2$) is given by the scalar field

$T = -(\alpha r^2/2) + b\cos\theta$ where α, b are constants and r is the distance from the centre of the concentric spheres to the point at which the temperature is considered.
At the same point the heat flow vector is given by the expression

$\mathbf{h} = \mathbf{a_r}\, c\, r$ in which c is a constant.

(the volume and the surface area of a sphere of radius r is $(4/3)\pi r^3$ and $4\pi r^3$ respectively)

(i) If $\mathbf{h} = -K\nabla T$ where K is the thermal conductivity; find K in terms of α, b and c
(ii) Verify the divergence theorem for the heat flow vector **h** for the region enclosed between the two spherical shells
(iii) Find the Laplacian of T (ΔT)
(iv) Comment about the solenoidal/irrotational nature of the heat flow vector **h**

(20 marks)

A2.
An incompressible fluid of constant density has a scalar velocity potential
$\Psi(x,y,z,t) = (-3x + z)(y/t)$

(i) Find the fluid velocity vector **v** associated with this potential, if $\mathbf{v} = -\nabla\Psi$
(ii) Find $\nabla \times \mathbf{v}$ and $\nabla \cdot \mathbf{v}$
(iii) If the condition of incompressibility is $\nabla \cdot \mathbf{v} = 0$, write the Laplace's equation for the fluid
(iv) Comment about the solenoidal/irrotational nature of the velocity vector **v**

(20 marks)

SECTION B:

Select **ONE** question

B1.

Suppose in a situation where within a very long region of cylindrical shape of radius b (i.e. the length of the cylinder $l >> b$) there exists a vector field $\mathbf{P} = P_0\, \mathbf{a}_z$ for $r<b$ and outside the cylindrical region, i.e. when $r>b$ the vector field $\mathbf{P} = 0$. r is the distance from the axis of the cylinder to a point on a plane, perpendicular to the cylindrical axis.

(i) By applying Stoke's theorem find a vector field **Q** inside and outside of the cylindrical region, such that $\mathbf{P} = \nabla \times \mathbf{Q}$

(ii) Verify by direct differentiation that $\nabla \times \mathbf{Q}$ is equal to the specified value of **P**

(20 marks)

B2.

Time varying electromagnetic fields in free space could be described by following Maxwell's equations:

$\nabla \times \mathbf{H} = \mathbf{J} + \partial \mathbf{D} / \partial t$ ....(1)

$\nabla \times \mathbf{E} = - \partial \mathbf{B} / \partial t$

$\nabla \bullet \mathbf{B} = 0$

$\nabla \bullet \mathbf{E} = 0$

Since $\mathbf{J} = \sigma \mathbf{E}$, $\mathbf{D} = \varepsilon_0 \varepsilon\, \mathbf{E}$ and $\mathbf{B} = \mu_0 \mu\, \mathbf{H}$ we could write equation (1) as follows:

$\nabla \times \mathbf{H} = \sigma \mathbf{E} + \varepsilon_0 \varepsilon\, \partial \mathbf{E} / \partial t$

$\quad = (\sigma + \varepsilon_0 \varepsilon\, \partial / \partial t)\, \mathbf{E}$ ....(2)

In the case of good conductors, the conductivity $\sigma >> \varepsilon_0 \varepsilon$ and we could write equation (2) as $\nabla \times \mathbf{H} = \sigma \mathbf{E}$ ....(3)

Using equation (3) and the remaining of Maxwell's equations, derive expressions which relates the time and the space rates of change, in the case of good conductors for

(i) the electric field **E**

(ii) the magnetic field **H**

(20 marks)

Hint:

$\nabla \times (\nabla \times A) = \nabla (\nabla \bullet A) - \nabla^2 A$

SECTION C:

Answer all questions

Read the attached article 'The vibrating-membrane problem - based on basic principles and simulations' by Hermann Härtel and Ernesto Martin and answer following questions.

C1.
Briefly explain in your own words the problem addressed in this article. Comment about the type of problem addressed (distributed/lumped parameter system) here.

(10 marks)

C2.
A student for Assignment 2 of ECX5241 derived the mathematical equation which explains small transverse vibrations of a thin rubber sheet – a membrane which has been stretched over a large horizontal frame (like a drumhead). Her answer is reproduced on pages 8 to 12 for your reference.
State all assumptions that should be done about the membrane at the problem statement and during derivation. What are the limitations of the mathematical model used to model the vibrating membrane?

(10 marks)

C3.
List in point form the steps you need to follow to solve the equation derived in question 2, using the 'pdetool' of MATLAB (10 marks)

C4.
What is the method used in MATLAB to solve partial differential equations? (5 marks)

C5.
How does the xyZET simulation tool mentioned in the article solve vibrating membrane problems? (5 marks)

C6.
According to the article, how does the simulation results compare with the theoretical results for vibrating membrane problems? (5 marks)

C7.
According to the authors of this article, what are the advantages of xyZET when compared with other simulation tools for the same purpose such as Mathematica, Derive or the one you have used 'pdetool' in MATLAB?

(15 marks)

0007

The vibrating-membrane problem - based on basic principles and simulations

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Abstract

Rectangular and circular membranes have been modelled as discrete arrays of mass points connected by massless springs. Based on Newton's principles and Hooke's law, the movement of such membranes has been simulated. All vibrational modes, as known from closed form solutions of the corresponding wave equations, can be excited, with deviations from theoretical values of no more than a few percent. This approach can be used to develop an intuitive understanding of vibrating membranes. The phenomenon of regular vibrational modes provides a suitable starting point for a thorough mathematical treatment.

In a more general sense this topic demonstrates the possibility that elasticity is no longer a matter of high mathematical demand. The true nature of the "rigid body" as an unrealistic but perfect model can convincingly be demonstrated.

1. Introduction

As has been demonstrated recently, the vibrating-membrane problem can be used as a rather appropriate example to demonstrate the power of computer algebra systems (CAS) like Axiom Maple, Mathematica, Derive etc. [1].

This approach, however, depends on a well-developed mathematical ability on the part of the learner and on his or her willingness to accept such an abstract and demanding path of explanation, where the solution of differential equations serves as a description of real world phenomena, in this case the vibrating modes of an elastic membrane.

In the following we would like to show that the same results can be achieved with much less mathematical effort and in a more direct fashion, based only on Newton's principles and linear elastic forces.

2. Theoretical Background

Our system consists of a plane membrane, in principle of any shape, homogeneously stretched by a tension T, given as force per unit length. The membrane has a mass μ per unit area and the boundary is clamped.

For small vibrations and in the absence of external forces the wave equation, describing the motion of the different points (coordinates *x, y* in the plane of the membrane), is [2]:

$$\nabla^2 s = \frac{\mu}{T}\cdot\frac{\partial^2 s}{\partial t^2} = \frac{1}{v^2}\cdot\frac{\partial^2 s}{\partial t^2}$$

∇^2 is the Laplace operator, $\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ in rectangular co-ordinates *x, y*, and $v \equiv \sqrt{T/\mu}$ is the velocity of the waves in the elastic membrane. We have denoted by *s(x,y,t)* the transverse displacement of any point relative to the position when the membrane is at rest.

For membranes held along the edge (s=0, as boundary condition), we have to find standing-wave solutions of the wave equation which have nodes along the boundary of the membrane. For simple shapes (rectangular or circular membranes), the standing wave solutions or normal modes of vibration are usually worked out using a set of curvilinear coordinates in which the edge of the membrane forms one of the coordinate axes. In many cases we can use separation of variables which simplifies the problem.

In the following the main characteristics of the modes for the rectangular and circular membranes are described. With our simulation tool xyZET [3] we can in principle experiment with membranes of any shape. The results in this article, however, are restricted to rectangular and circular geometries which allows us to compare our simulated results with theoretical solutions of the related wave equation.

Rectangular Membrane (borders fixed: s=0 in x=0,a and y=0,b)

By separating the variables (s=X(x)Y(y)exp(iωt), the standing wave modes for this case can be expressed as follows:

$\sin(k_x x)\cdot\sin(k_y y)$, multiplied by a harmonic time dependence $\sin(\omega_0 t)$, where the resonance frequency, ω_0, will depend on the mode of vibration (values of k_x, k_y)

$$\left(\frac{\omega_0}{v}\right)^2 = k_x{}^2 + k_y{}^2 .$$

The boundary conditions require that k_x, k_y can have only the following values: $k_x = \frac{m\pi}{a}$, $k_y = \frac{n\pi}{b}$, where m,n (the mode indexes) can take only integer values.

The resonance frequency for this (m,n) mode will be

$$\left(\frac{\omega_{mn}}{v}\right)^2 = k_{xm}{}^2 + k_{yn}{}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

with $v \equiv \sqrt{T/\mu}$.

Circular Membrane (border fixed: s=0, for r=a)

In this case we can use polar co-ordinates (r,θ). The spatial part of the wave function will be of the form R(r)Θ(θ). The boundary conditions will act specifically on R(r) which will be a Bessel function $J_m(kr)$ with zeros at well known (tabulated) values x_{mn} (m for the function and n for the nth. zero).

This leads to the relation $k_{mn}a=x_{mn}$ to force a zero at r=a, the radius of the membrane. This results in the following relation for computing the angular frequency associated with the different modes:

$\omega_{mn} = \frac{x_{mn}v}{a}$, where v is the velocity of the wave in the membrane.

The solution of our problem for the mode (m,n) is, basically, of the form:

$$s(r,\theta,t) = J_m(k_{mn}r)\cos(m\theta)\cos(\omega_{mn}t) .$$

A dependence with sin(mθ) is also possible, giving rise to the existence of 2 degenerate modes for each m (except for m=0). In general, a linear combination of both modes will be excited.

3. The simulation program xyZET

At IPN, a simulation program, named xyZET, has been developed whose key feature is the visualization of interacting objects in 3d [1] [3] [4]. The effects of all classical forces can be simulated.

The implemented algorithm is force based. For each single particles of all those placed within the cube, the sum of all applied forces is determined. By integrating Newtons second law stepwise, the acceleration, the change in velocity and the resulting displacement is calculated and displayed by deleting and redrawing the particles at its new positions.

Figure 1 shows circular and rectangular membranes as modelled in xyZET, where the particles at the border are fixed and all particles are connected by springs with their nearest neighbours.

Placing particles and connecting particles by springs is done by repetitive mouse clicks. The cube which surrounds the objects can be rotated to show the system from different perspectives.

All relevant parameters such as charge, mass, spring constant and spring length can be set and an external electric field can be simulated, changing in time and with variable intensity, period and direction.

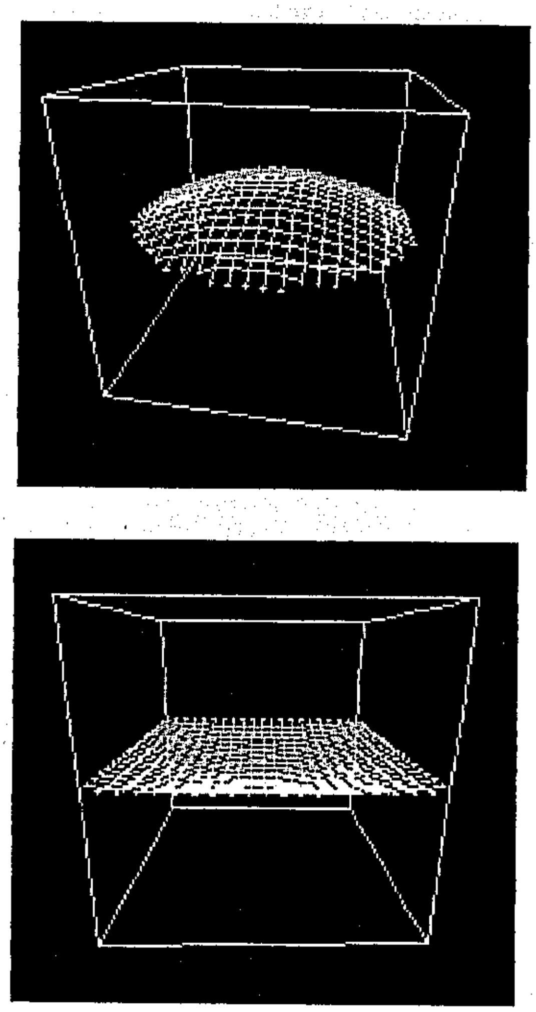


Figure 1 Membranes as modelled within xyZET

4. Experiments

To compare the simulation results from xyZET with those predicted theoretically, we have experimented with rectangular and circular membranes.

To do this, the mechanical characteristics of the membrane like tension T, and density μ, have to be determined. This information can be obtained from data available within xyZET.

1. Demo version download: http://www.ipn.uni-kiel.de/english/projekte/a7/a7.1/xyzet/mainpage_e.html

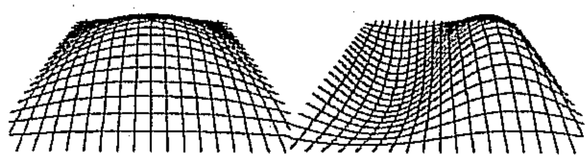
Once these values have been measured, the wave velocity, v, in the membrane, the eigenvalue and eigenfunction (resonant frequency and spatial distribution) for every mode can be computed as shown in the previous paragraph.

Results for a rectangular membrane

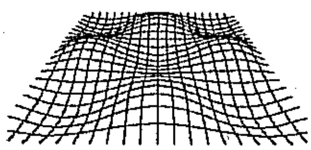
The membrane we used was made up of 21x21 particles, homogeneously distributed in a rectangular grid. From the measured values for tension T and density μ the wave velocity for mode 11, 21 and 33 was computed as well as the resonant frequencies ω_0 of the different modes.

By charging a few single particles, positioned at symmetry points of the expected mode and applying an external electric alternating field with ω_0 as frequency, the corresponding mode can be excited.

The agreement between the calculated resonant frequencies and the one measured with xyZET is between 3 and 5% for the lowest order modes. The spatial distribution of some of these modes are shown in Figure 2.



mode 11 mode 21

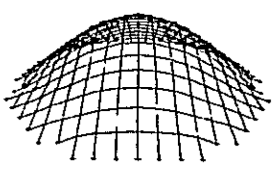


mode 33

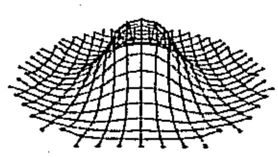
Figure 2 Display of the vibrating membrane for different modes

Results for a circular membrane

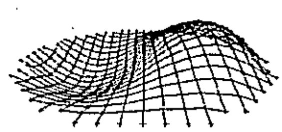
The membrane we used was again made up of 21x21 particles, homogeneously distributed over a circular area. The modes displayed in Figure 3 were excited and its velocity compared with the theoretical values.



mode 01



mode 02



mode 11

Figure 3 Display of a circular membrane vibratin different modes

The differences between the simulated and theore results was always less than 6%.

5. Discussion

Didactical aspects

The topic "vibrating membranes" is a specific one primarily only of interest for a specialised branch of e neering. For lectures in physics this topic is usually aside due to the high demands of mathematics needed the experimental difficulties to demonstrate the regu ties of different vibrating modes.

Both these limiting factors have vanished. The pow modern computers allows to demonstrate all kinds of ularly or irregularly shaped membranes in their diffe vibrating modes in an effortless way. The question th fore has to be posed if this topic has some general did cal value and relevance.

We see two aspects: 1. With the support of mo computers the behaviour of extended elastic objects easily be integrated in the physics curriculum. Nowa extended objects are most of the time treated as r which implies certain problems [5]. The model of the body is an artificial one, neglecting internal processes relying on non-causal distributions of forces. The t ment of extended bodies in physics could therefor enriched if such objects would not only be presente rigid but also as elastic - their real and only nature.

2. The whole is more than the sum of its parts. basic statement can be visualised in a rather convin and surprising way by applying our method, descr above. When exciting for instance a single point in the tre of the rectangular membrane (fig 1 below) with an a trary frequency, some irregular vibrations of the comp membrane are displayed and regular pattern, if at all ble, show up only for short moments in time. But if the quency is one of the eigenvalues of the membrane, the vibrations around the excited particle at the centre slo but irresistibly transform to a vibrating mode which c trols every single particle of the membrane in a coo nated way. Such a mode is a property of the complete tem. It cannot be derived from properties of its parts a is more than the sum of all these individual properties.

We dare to mention that our swinging membranes not only correctly represented but are nice and attractiv look at. This fact cannot be a substitute for learning pl ics, but it will never harm and may be more important motivation than often acknowledged.

Besides these general aspects a more specific p deserves to be mentioned: the good agreement betwe simulated membrane, modelled as a system of disc parts and the theory, based on a continuous distributio matter. This aspect is discussed in the following paragra

Explanation of the difference between theory and si ulation

Results from theory and our simulation differ due number of factors.

First, the measurement of the resonance frequency carries an experimental error due to the method used. Resonance is detected by observing the shape of the space distribution of the vibrating elastic plane, and although this shape and the associated amplitude are very sensitive to frequency variations, we estimate an error in this measurement of the order of 1%.

Second, our model is a discrete one while the theory is based on a continuous mass distribution. Since at vibrating modes of higher order the spatial distribution of mass varies more strongly, the difference between the "continuous" theory and the discrete model should increase with modes of higher order. For a linear string, modelled by elastically connected mass points with the same spatial distribution as our plane, we computed this expected tendency with a maximum deviation of 1% for the 3rd. order mode (for a string made up of 21 particles).

A third reason for the difference between theory and experiment can be found in the fact that the theory does not take into account the variation of tension in time within any swinging plane. Such an idealization is valid only for rather small amplitudes. The theory also does not take into account the fact that for larger amplitudes the displacements do not only occur perpendicularly but also to a small degree in parallel to the plane. Since we need larger amplitudes to measure the resonance frequency, it cannot be expected that our simulated model results in the same values as those derived from the idealized theory.

Finally, the membranes we used in our simulation with circular planes did not have a precisely circular shape. This also may explain some of the difference between theory and experiment.

Basic laws and numerical solutions as added value

Traditional methods for teaching topics like "oscillations and waves" are characterized by doing experiments and then solving the corresponding differential equations. Experiments should be carried out whenever possible, and a thorough theoretical treatment is necessary. Depth and direction of this theoretical treatment, however, are open to discussion.

Besides looking only for closed form solutions of wave equations in 2 dimensions (Bessel functions in cylindrical coordinates), a more direct and much simpler path is now opened by starting from Newton's basic principles and Hooke's law and by looking for the corresponding numerical solutions, visualized on a computer screen.

Furthermore, this approach allows for a broad spectrum of exploratory actions. Direct feedback is received when changing the shape of the plane or internal parameters such as mass distribution and tension. This offers the possibility of building up an intuitive knowledge base about the behaviour of membranes as a starting point for the mathematical treatment.

We therefore see this approach not as an alternative but as an enrichment to the traditional method. The relation between cause, condition and effect is shown in a more direct manner and is offered for experimental exploration. Furthermore the comparison between our numerical solutions and the closed form solutions offers the opportunity to develop methodological knowledge of higher order.

6. References

[1] R. Portugal, L. Golebiowski, D. Frenkel, *Oscillation of membranes using computer algebra*, Am. J. Phys. 67 (6), 534-537 (June 1999).

[2] J. C. Slater and N. H. Frank, *Mechanics*, McGraw-Hill, NY, 1947.

[3] H. Härtel, *xyZET - A Simulation Program for Physics Teaching*. Journal for Science and Education Technology, 9, 3, 275-286, 2000

[4] H. Härtel, M. Lüdke, *3D-Simulations of Interacting Particles*, Computing in Science & Engineering, 4, 87-90, 2000

[5] H. Härtel, From simple to complex. But what is simple and for whom? This edition.

Deflection of a Stretched Membrane.

Here we will be considering **small out-of-the plane deflections** (deflections perpendicular to the initial plane of the membrane) of a **circular membrane**... A drawing of a circular membrane subjected to uniform tension is given below...

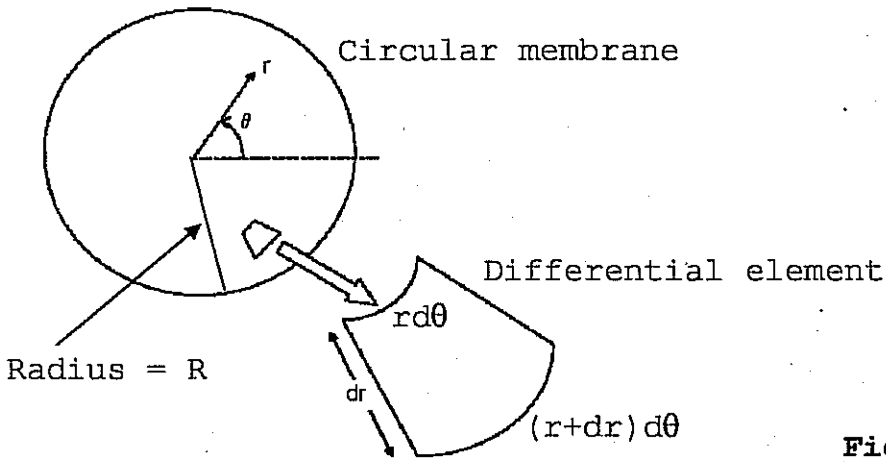


Figure - 1

- The deflections of the planar membrane from its equilibrium position will be considered as **z(r,θ,t)**.
- To avoid complexity a force acting normal to the surface of the membrane is considered here.
- Such a force, causes the membrane to bulge in its direction and reach a new equilibrium state as given in 'Figure – 2'.

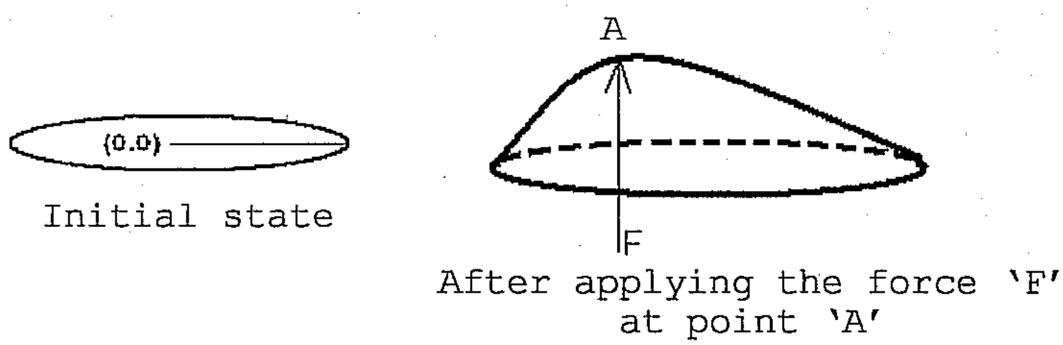


Figure - 2

The only restoring force present in the membrane is the **in-plane tension**

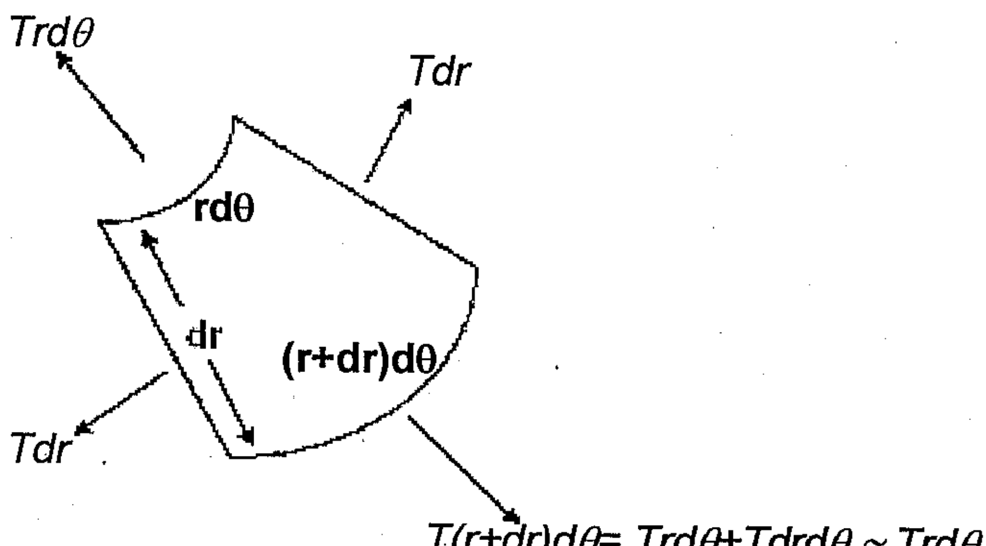


Figure - 3

Using **Newton's Second Law** on the above differential element we can obtain the equations governing the deflection of the above considered membrane

Deriving the equations governing the behavior of the membrane...

First, the forces in a plane 'θ = **constant' are considered as in 'Figure-4'.**

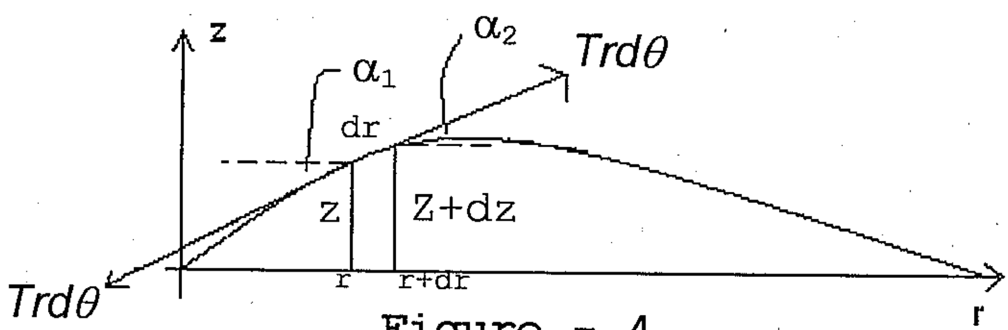


Figure - 4

Small deflections are considered,

Hence ; $r\sin\alpha_1 \approx r\tan\alpha_1 = r\frac{\delta z}{\delta r} = r\left.\frac{\partial z}{\partial r}\right|_r$ → (2

$$r\sin\alpha_2 \approx r\tan\alpha_2 = r\frac{\delta z}{\delta r} = r\left.\frac{\partial z}{\partial r}\right|_{r+dr}$$

→ (3

Expanding equation (3) around 'r' using **Taylor series**, we obtain the approximate value for (3)as...

$$r\left.\frac{\partial z}{\partial r}\right|_{r+dr} \approx r\left.\frac{\partial z}{\partial r}\right|_r + (r+dr-r)\frac{\partial}{\partial r}\left.\left(r\frac{\partial z}{\partial r}\right)\right|_r + \frac{dr^2}{2!}\frac{\partial^2}{\partial r^2}\left.\left(r\frac{\partial z}{\partial r}\right)\right|_r + h.o.t$$

;

h.o.t-higher order terms
Higher order terms are neglected, only 1st order term of 'dr' is taken in to account ...
Then;

$$r\sin\alpha_1 \approx r\tan\alpha_1 = r\frac{\partial z}{\partial r}\bigg|_{r+dr} \approx r\frac{\partial z}{\partial r}\bigg|_r + dr\frac{\partial}{\partial r}\left(r\frac{\partial z}{\partial r}\right)\bigg|_r \quad \rightarrow (4)$$

Substituting (2)&(4)in (1)...

$$\uparrow \sum F_1 = Td\theta\left[-r\sin\alpha_1 + r\sin\alpha_2\right]$$

$$\uparrow \sum F_1 = Td\theta\left[-r\frac{\partial z}{\partial r}\bigg|_r + r\frac{\partial z}{\partial r}\bigg|_r + dr\frac{\partial}{\partial r}\left(r\frac{\partial z}{\partial r}\right)\bigg|_r\right] = Td\theta\left[dr\frac{\partial}{\partial r}\left(r\frac{\partial z}{\partial r}\right)\bigg|_r\right]$$

$$\uparrow F_1 = T\frac{\partial}{\partial r}\left(r\frac{\partial z}{\partial r}\right)\bigg|_r d\theta dr \quad \rightarrow (5)$$

The forces in a plane '**r= constant**' which is normal to 'θ= **constant**', are considered as in '**Figure-5**'.

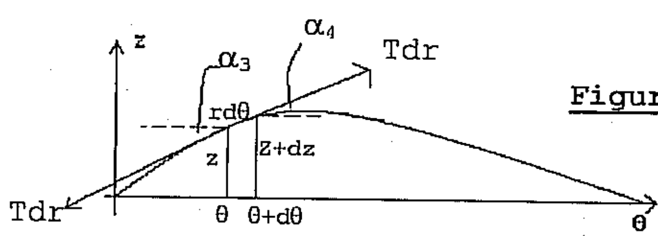


Figure - 5

$$\uparrow \sum F_2 = -Tdr(\sin\alpha_3) + Tdr(\sin\alpha_4) \quad \rightarrow (6)$$

Small deflections are considered,

Hence ; $$\sin\alpha_3 \approx \tan\alpha_3 = \frac{\delta z}{r\delta\theta} = \frac{\partial z}{r\partial\theta}\bigg|_\theta \quad \rightarrow (7$$

$$\sin\alpha_4 \approx \tan\alpha_4 = \frac{\delta z}{r\delta\theta} = \frac{\partial z}{r\partial\theta}\bigg|_{\theta+d\theta} \quad \rightarrow (8$$

As earlier getting the 'Taylor series expansion' of (8) around 'θ' and neglecting its higher order terms, we obtain...

As earlier getting the 'Taylor series expansion' of (8) around 'θ' and neglecting its higher order terms, we obtain...

$$\sin\alpha_4 \approx \tan\alpha_4 = \left.\frac{\partial z}{r\partial\theta}\right|_{\theta+d\theta} \approx \left.\frac{\partial z}{r\partial\theta}\right|_{\theta} + d\theta\frac{\partial}{\partial\theta}\left(\frac{\partial z}{r\partial\theta}\right)\Big|_{\theta} = \left.\frac{\partial z}{r\partial\theta}\right|_{\theta} + d\theta\left.\frac{\partial^2 z}{r\partial\theta^2}\right|_{\theta} \quad \rightarrow (9)$$

Substituting (7)&(9) in (6)...

$$\uparrow \sum F_2 = Tdr[-(\sin\alpha_3)+(\sin\alpha_4)] = Tdr\left[-\left.\frac{\partial z}{r\partial\theta}\right|_{\theta} + \left.\frac{\partial z}{r\partial\theta}\right|_{\theta} + d\theta\left.\frac{\partial^2 z}{r\partial\theta^2}\right|_{\theta}\right]$$

$$\uparrow \sum F_2 = T\left.\frac{\partial^2 z}{r\partial\theta^2}\right|_{\theta} d\theta dr \quad \rightarrow (10)$$

Consider the mass per unit area of the membrane as 'σ'.
Then the mass of the differential element considered ...

m = σ(dr)(rdθ)

Applying '**Newtons Second Law**' to the differential element...

Σ F = ma

F-force applied **m**-mass of the element **a**-acceleration

F = F1 + F2 = [σ(dr)(rdθ)](a)

Substituting values for F1 & F2 from (5) & (10)

$$F = T\frac{\partial}{\partial r}\left(r\frac{\partial z}{\partial r}\right)\Big|_r d\theta dr + T\left.\frac{\partial^2 z}{r\partial\theta^2}\right|_{\theta} d\theta dr$$

$$F = \frac{T}{r}\frac{\partial}{\partial r}\left(r\frac{\partial z}{\partial r}\right)\Big|_r rd\theta dr + \frac{T}{r^2}\left.\frac{\partial^2 z}{\partial\theta^2}\right|_{\theta} rd\theta dr = \sigma rdrd\theta\left(\frac{\partial^2 z}{\partial t^2}\right)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial z}{\partial r}\right)\Big|_r + \frac{1}{r^2}\left.\frac{\partial^2 z}{\partial\theta^2}\right|_{\theta} = \frac{\sigma}{T}\left(\frac{\partial^2 z}{\partial t^2}\right)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial z}{\partial r}\right)\Big|_r + \frac{1}{r^2}\left.\frac{\partial^2 z}{\partial\theta^2}\right|_{\theta} = \frac{\sigma}{T}\left(\frac{\partial^2 z}{\partial t^2}\right) = \frac{1}{r}\left.\frac{\partial z}{\partial r}\right|_r + \left.\frac{\partial^2 z}{\partial r^2}\right|_r + \frac{1}{r^2}\left.\frac{\partial^2 z}{\partial\theta^2}\right|_{\theta}$$

$$\nabla^2 z = \frac{1}{c^2}\left(\frac{\partial^2 z}{\partial t^2}\right) \rightarrow (11) \qquad c = \sqrt{\frac{T}{\sigma}}$$

Similarly for a rectangular membrane we can obtain the governing equation in rectangular coordinates as...

$$z_{xx} + z_{yy} = \frac{1}{c^2} z_{tt}$$

$$c = \sqrt{\frac{T}{\sigma}}$$

$$\nabla^2 z = \frac{1}{c^2} z_{tt}$$

This is considered as the 'wave equation'

VECTOR RELATIONS

DIFFERENTIAL ELEMENTS OF VECTOR LENGTH

$$d\mathbf{l} = \begin{cases} \mathbf{a}_x\, dx + \mathbf{a}_y\, dy + \mathbf{a}_z\, dz \\ \mathbf{a}_\rho\, d\rho + \mathbf{a}_\phi \rho\, d\phi + \mathbf{a}_z\, dz \\ \mathbf{a}_r\, dr + \mathbf{a}_\theta r\, d\theta + \mathbf{a}_\phi r \sin\theta\, d\phi \end{cases}$$

DIFFERENTIAL ELEMENTS OF VECTOR AREA

$$d\mathbf{s} = \begin{cases} \mathbf{a}_x\, dy\, dz + \mathbf{a}_y\, dx\, dz + \mathbf{a}_z\, dx\, dy \\ \mathbf{a}_\rho \rho\, d\phi\, dz + \mathbf{a}_\phi\, d\rho\, dz + \mathbf{a}_z\, \rho\, d\rho\, d\phi \\ \mathbf{a}_r r^2 \sin\theta\, d\theta\, d\phi + \mathbf{a}_\theta r \sin\theta\, dr\, d\phi + \mathbf{a}_\phi r\, dr\, d\theta \end{cases}$$

DIFFERENTIAL ELEMENTS OF VOLUME

$$dv = \begin{cases} dx\, dy\, dz \\ \rho\, d\rho\, d\phi\, dz \\ r^2 \sin\theta\, dr\, d\theta\, d\phi \end{cases}$$

VECTOR OPERATIONS—RECTANGULAR COORDINATES

$$\nabla\alpha = \mathbf{a}_x \frac{\partial \alpha}{\partial x} + \mathbf{a}_y \frac{\partial \alpha}{\partial y} + \mathbf{a}_z \frac{\partial \alpha}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \mathbf{a}_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \mathbf{a}_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \mathbf{a}_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla^2\alpha = \frac{\partial^2 \alpha}{\partial x^2} + \frac{\partial^2 \alpha}{\partial y^2} + \frac{\partial^2 \alpha}{\partial z^2} \equiv \nabla \cdot \nabla\alpha$$

$$\nabla^2\mathbf{A} = \mathbf{a}_x \nabla^2 A_x + \mathbf{a}_y \nabla^2 A_y + \mathbf{a}_z \nabla^2 A_z \equiv \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A})$$

VECTOR OPERATIONS—CYLINDRICAL COORDINATES

$$\nabla\alpha = \mathbf{a}_\rho \frac{\partial \alpha}{\partial \rho} + \mathbf{a}_\phi \frac{1}{\rho}\frac{\partial \alpha}{\partial \phi} + \mathbf{a}_z \frac{\partial \alpha}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \mathbf{a}_\rho\left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \mathbf{a}_\phi\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right) + \mathbf{a}_z \frac{1}{\rho}\left(\frac{\partial}{\partial \rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi}\right)$$

$$\nabla^2 \alpha = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \alpha}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \alpha}{\partial \phi^2} + \frac{\partial^2 \alpha}{\partial z^2}$$

$$\nabla^2 \mathbf{A} = \mathbf{a}_\rho\left(\nabla^2 A_\rho - \frac{2}{\rho^2}\frac{\partial A_\phi}{\partial \phi} - \frac{A_\rho}{\rho^2}\right) + \mathbf{a}_\phi\left(\nabla^2 A_\phi + \frac{2}{\rho^2}\frac{\partial A_\rho}{\partial \phi} - \frac{A_\phi}{\rho^2}\right) + \mathbf{a}_z \nabla^2 A_z$$

Vector Operations – Spherical coordinates

$$\nabla \alpha = a_r \cdot \frac{\partial \alpha}{\partial r} + a_\theta \frac{1}{r} \frac{\partial \alpha}{\partial \theta} + a_\phi \frac{1}{r \sin\theta} \cdot \frac{\partial \alpha}{\partial \phi}$$

$$\nabla \cdot A = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \cdot \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times A = \frac{1}{r^2 \sin\theta}\begin{vmatrix} a_r & r \cdot a_\theta & r\sin\theta \cdot a_\phi \\ \partial/\partial r & \partial/\partial \theta & \partial/\partial \phi \\ A_r & r \cdot A_\theta & r\sin\theta \cdot A_\phi \end{vmatrix}$$

$$\nabla^2 \alpha = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \alpha}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \cdot \frac{\partial \alpha}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \alpha}{\partial \phi^2}$$

$$\nabla \times \nabla \times A = \nabla(\nabla \cdot A) - \nabla^2 A$$

$$\frac{d}{dx}(\sin ax) = a \cos x$$

$$\frac{d}{dx}(\cos ax) = -a \sin x$$

$$\int \sin ax \cdot dx = -(\cos ax)/a$$

$$\int \cos ax \cdot dx = \sin ax / a$$