The Open University of Sri Lanka
Department of Electrical and Computer Engineering
Bachelor of Technology (Engineering)
ECX5241– Distributed Parameter Systems
Final Examination - 2010/2011
Duration : Three hours.

Date: 13th March 2011	Time: 0930-1230 hrs

The paper contains three sections A, B & C. Answer **all questions** in section A and **ONE** question each from sections B & C.
Clearly show all the steps and state all the assumptions made.

SECTION A

1) Questions in Section A, question 1 are based on a note compiled incorporating the lecture notes *"Wave Propagation, Fall, 2006 MIT, notes by C.C. Mei"* given on pages 4 to 6. Read these notes and answer the following questions.

i. Describe in your own words the scenario being modeled in this note. [4 *marks*]

ii. Name three distributed parameters encountered in this scenario. State whether each of them are vector or scalar quantities. [6 *marks*]

iii. State the assumptions made. [4 *marks*]

iv. a. Write an equation for the conservation of mass of volume 'dv'. [10 *marks*]

b. Hence deduce equation (1). [5 *marks*]

v. a. Write an equation for the momentum balance of volume 'dv' using expressions (2) to (5). [5 *marks*]

b. Using the above and equation (1) deduce equation (6). [10 *marks*]

vi. Obtain an expression for 'C' in equation (9). [10 *marks*]

vii. a. Second order partial differential equations can be classified as elliptic, parabolic and hyperbolic equations. According to this classification what is the type equations in (14), (15) & (16)? Justify. [3 *marks*]

b. State three other phenomena which you encountered in your course, which yield in the same form of governing equations. [3 *marks*]

[Total : 60 *marks*]

SECTION B

2) Let $\underline{F} = (1-y^2)\underline{i}$ be defined in the region $\{-\infty < x < \infty, -1 \leq y \leq 1\}$. $\underline{F}$ is the velocity field of water flowing in a channel occupying the above region. A peddle wheel is to be placed in the x-y plane as shown in figure Q2.

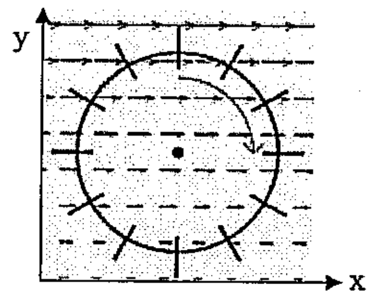

Figure Q2

i. In what region should the paddle wheel be placed,
   a) To rotate clock wise?
   b) To rotate anti clockwise?
   c) For no rotation?

   Justify your answers.
ii. Find $\nabla \circ (\nabla \times \underline{F})$.
iii. Is $\underline{F}$ an incompressible vector field? Justify your answer.
iv. Find $\nabla \times (\nabla \times \underline{F})$.

[20 *marks*]

3) The amount of heat energy leaving the boundary surface S of a region R in unit time can be given by; $\oiint_S \underline{Q}.\underline{n}dx.dy$. Where Q is the heat flux and n is the unit normal vector to surface S.

Also $\underline{Q} = -k\nabla T$ and $dE = -\rho C dT$.

And the energy equation can be given in the integral form as;

$$\oiiint_V \rho \frac{\partial E}{\partial t}(dx.dy.dz) = \oiint_S \underline{Q}.\underline{n}dx.dy$$

Where k is conductivity, ρ is density (kg/m^3), E is the internal energy, C is specific heat capacity and T is temperature.

i. Is Q a conservative field? Comment.
ii. Write an expression for $\oiint_S \underline{Q}.\underline{n}dx.dy$ using 'Divergence theorem'.
iii. Hence obtain the partial differential equation governing the heat flow.

[20 *marks*]

SECTION C

4) Imagine a straight horizontal pipe of cross sectional area $A\,m^2$, filled by a motionless liquid, and a dye diffusing through the liquid. The dye moves from regions of higher concentration to the regions of lower concentration. The mass flow rate is proportional to the concentration gradient of the dye. Let u(x,t) be the concentration (mass per unit volume)of the dye at position x of the pipe at time t. Derive the governing *P.D.E.* for the concentration of the dye in the liquid.

Note:
No chemical mixing of the dye and the liquid is present.

[20 *marks*]

5) Derive and show that the equation governing the torsional (twist) vibrations of a circular shaft results,

$$\frac{\partial^2\theta}{\partial t^2}=\frac{G}{\rho}\frac{\partial^2\theta}{\partial x^2},$$ Where ρ is mass density and G is a constant.

The basic laws governing the movement states that $I\alpha = T$, where α is the angular acceleration, T is the torque, and I is the mass moment of inertia.

Note:

$T=\frac{GJ\theta}{L}$, Where θ is angle of twist of a shaft of length L. J and G are constants.

$I=\frac{JM}{A}$, Where M is the mass of the shaft and A is the cross sectional area.

[20 *marks*]

Wave Propagation in Arteries

We shall examine the pulsating flow of blood in an artery whose wall is thin and elastic. As a first exercise let us assume that there is only pulsation but no net flux. Because of the pressure gradient in the blood, the artery wall must deform. The elastic restoring force in the wall makes it possible for waves to propagate.

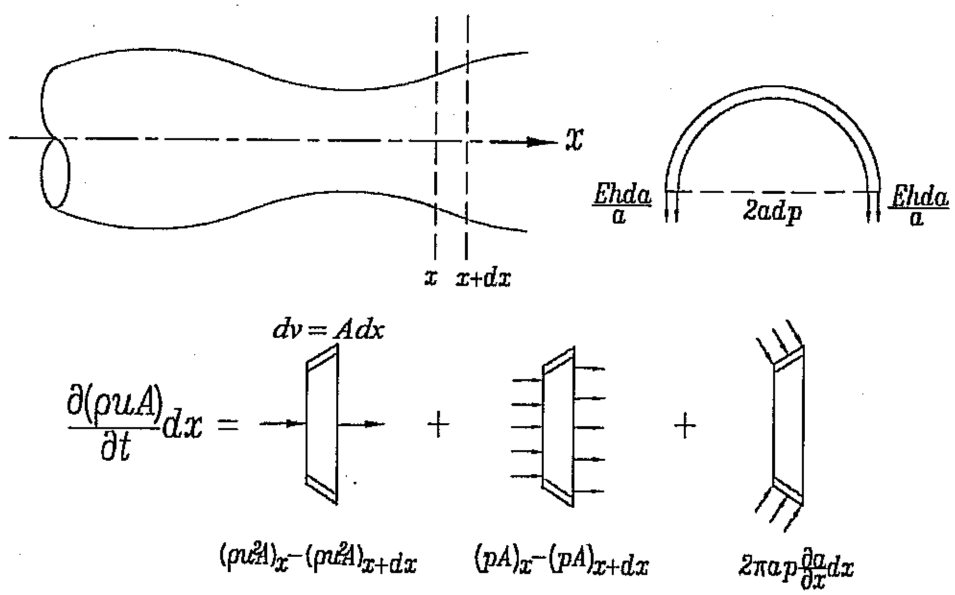

Figure 1 : Forces on the artery wall

The artery radius a(x,t) varies from the constant mean a_o in time and along the artery (in x). Let the local cross sectional area be $A = \pi a^2$, and the averaged velocity be u(x,t) and averaged blood pressure P(x,t). Consider a fixed geometrical volume 'dv' between x and x + dx, through which fluid moves in and out. The density of blood ρ is assumed to be constant. Conservation of mass states that the rate of change of mass in dv equals the rate of mass in flow minus the rate of mass out flow from dv. Hence we can deduce,

$$\frac{\partial A}{\partial t}+\frac{\partial uA}{\partial x}=0 \qquad (1)$$

Next the momentum balance will be considered. The time rate of momentum change in the volume is equal to the sum of rate of momentum influx through the two ends of the volume and the pressure forces acting on all boundaries. The time rate of momentum change is,

$$\frac{\partial u.\rho A}{\partial t}dx \qquad (2)$$

The rate of momentum influx is

$$-[(\rho uA)u\,|_x-(\rho uA)u\,|_{x+dx}] \qquad (3)$$

The net pressure force on the two boundaries at x and x+dx is

$$-[(PA)\,|_x-(PA)\,|_{x+dx}] \qquad (4)$$

While that on the sloping wall is

$$2\pi a P\frac{\partial a}{\partial x}dx = Pdx\frac{\partial \pi a^2}{\partial x} = Pdx\frac{\partial A}{\partial x} \quad (5)$$

Balancing the momentum and after making use of equation mass conservation (1), we get,

$$\rho\left\{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right\} = -\frac{\partial P}{\partial x} \quad (6)$$

Let the pressure outside the artery be constant, say zero. The change in the tube radius must be caused by the change in blood pressure. Referring to Figure 1, the elastic strain due to the lengthening of the circumference is $2\pi(da)/2\pi a = (da)/a$. Let h be the artery wall thickness, assumed to be much smaller than a, and Young's modulus E. The change in elastic force is 2Eh(da)/a which must be balanced by the change in pressure force 2a(dp) , i.e.,

$$\frac{2Eh(da)}{a} = 2a(dp)$$

Which implies,

$$\frac{\partial P}{\partial a} = \frac{Eh}{a^2}$$

$$\pi a^2 = A \Rightarrow da = dA/2\pi a$$

$$\frac{\partial P}{\partial A} = \frac{\sqrt{\pi}Eh}{2A^{3/2}} \quad (7)$$

Pressure increases with the tube radius, but the rate of increase is smaller for larger radii. Upon integration we get the equation of state

$$P - P_o = -\frac{Eh}{a} = -\frac{\sqrt{\pi}Eh}{2\sqrt{A}} \quad (8)$$

Using the results in equation (7), equation (6) may now be rewritten as,

$$A\left\{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right\} = -C^2\frac{\partial A}{\partial x} \quad (9)$$

C has the dimensions of velocity.

In view of (7), equations (1) and (9) are a pair of nonlinear equations for the two unknowns, u and S.

Linearization:

For infinitesimal amplitudes we can linearize these equations. Let $a = a_o + a'$ with $a' << a_o$ then the (1) becomes, to the leading order,

$$\frac{\partial a'}{\partial t} + \frac{a_o}{2}\frac{\partial u}{\partial x} = 0 \quad (10)$$

The linearized momentum equation is

$$\rho\frac{\partial u}{\partial t} + \frac{\partial P}{\partial x} = 0 \quad (11)$$

The linearized form of (7) is

$$\frac{Eh(da')}{a_o^2} = (dp) \qquad (12)$$

Which can be used in (11) to get,

$$\rho\frac{\partial u}{\partial t} = -\frac{Eh}{a_o^2}\frac{\partial a'}{\partial x} \qquad (13)$$

Finally (10) and (13) can be combined to give ,

$$\frac{\partial^2 a'}{\partial t^2} = c_o^2\frac{\partial^2 a'}{\partial x^2} \qquad (14)$$

Where, $c_o = \sqrt{\frac{Eh}{2\rho a_o}}$

Alternately one can eliminate a' to get an equation for u,

$$\frac{\partial^2 u}{\partial t^2} = c_o^2\frac{\partial^2 u}{\partial x^2} \qquad (15)$$

Because of (12), the dynamic pressure is governed also by,

$$\frac{\partial^2 P}{\partial t^2} = c_o^2\frac{\partial^2 P}{\partial x^2} \qquad (16)$$

All unknowns are governed by the same equation due to linearity and the fact that all coefficients are constants.

VECTOR RELATIONS

DIFFERENTIAL ELEMENTS OF VECTOR LENGTH

$$d\mathbf{l} = \begin{cases} \mathbf{a}_x\, dx + \mathbf{a}_y\, dy + \mathbf{a}_z\, dz \\ \mathbf{a}_\rho\, d\rho + \mathbf{a}_\phi \rho\, d\phi + \mathbf{a}_z\, dz \\ \mathbf{a}_r\, dr + \mathbf{a}_\theta r\, d\theta + \mathbf{a}_\phi r \sin\theta\, d\phi \end{cases}$$

DIFFERENTIAL ELEMENTS OF VECTOR AREA

$$ds = \begin{cases} \mathbf{a}_x\, dy\, dz + \mathbf{a}_y\, dx\, dz + \mathbf{a}_z\, dx\, dy \\ \mathbf{a}_\rho \rho\, d\phi\, dz + \mathbf{a}_\phi\, d\rho\, dz + \mathbf{a}_z\, \rho\, d\rho\, d\phi \\ \mathbf{a}_r r^2 \sin\theta\, d\theta\, d\phi + \mathbf{a}_\theta r \sin\theta\, dr\, d\phi + \mathbf{a}_\phi r\, dr\, d\theta \end{cases}$$

DIFFERENTIAL ELEMENTS OF VOLUME

$$dv = \begin{cases} dx\, dy\, dz \\ \rho\, d\rho\, d\phi\, dz \\ r^2 \sin\theta\, dr\, d\theta\, d\phi \end{cases}$$

VECTOR OPERATIONS—RECTANGULAR COORDINATES

$$\nabla\alpha = \mathbf{a}_x \frac{\partial \alpha}{\partial x} + \mathbf{a}_y \frac{\partial \alpha}{\partial y} + \mathbf{a}_z \frac{\partial \alpha}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \mathbf{a}_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \mathbf{a}_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \mathbf{a}_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla^2\alpha = \frac{\partial^2 \alpha}{\partial x^2} + \frac{\partial^2 \alpha}{\partial y^2} + \frac{\partial^2 \alpha}{\partial z^2} \equiv \nabla \cdot \nabla\alpha$$

$$\nabla^2\mathbf{A} = \mathbf{a}_x \nabla^2 A_x + \mathbf{a}_y \nabla^2 A_y + \mathbf{a}_z \nabla^2 A_z \equiv \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A})$$

VECTOR OPERATIONS—CYLINDRICAL COORDINATES

$$\nabla\alpha = \mathbf{a}_\rho \frac{\partial \alpha}{\partial \rho} + \mathbf{a}_\phi \frac{1}{\rho}\frac{\partial \alpha}{\partial \phi} + \mathbf{a}_z \frac{\partial \alpha}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \mathbf{a}_\rho\left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \mathbf{a}_\phi\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right) + \mathbf{a}_z \frac{1}{\rho}\left(\frac{\partial}{\partial \rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi}\right)$$

$$\nabla^2\alpha = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \alpha}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \alpha}{\partial \phi^2} + \frac{\partial^2 \alpha}{\partial z^2}$$

$$\nabla^2\mathbf{A} = \mathbf{a}_\rho\left(\nabla^2 A_\rho - \frac{2}{\rho^2}\frac{\partial A_\phi}{\partial \phi} - \frac{A_\rho}{\rho^2}\right) + \mathbf{a}_\phi\left(\nabla^2 A_\phi + \frac{2}{\rho^2}\frac{\partial A_\rho}{\partial \phi} - \frac{A_\phi}{\rho^2}\right) + \mathbf{a}_z \nabla^2 A_z$$

Vector Operations – Spherical coordinates

$$\nabla\alpha = a_r \cdot \frac{\partial \alpha}{\partial r} + a_\theta \frac{1}{r}\frac{\partial \alpha}{\partial \theta} + a_\phi \cdot \frac{1}{r\sin\theta}\cdot\frac{\partial \alpha}{\partial \phi}$$

$$\nabla \cdot A = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \cdot \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times A = \frac{1}{r^2\sin\theta}\begin{vmatrix} a_r & r \cdot a_\theta & r\sin\theta \cdot a_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ A_r & r \cdot A_\theta & r\sin\theta \cdot A_\phi \end{vmatrix}$$

$$\nabla^2\alpha = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \alpha}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \cdot \frac{\partial \alpha}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 \alpha}{\partial \phi^2}$$

$$\nabla \times \nabla \times A = \nabla(\nabla \cdot A) - \nabla^2 A$$

$$\frac{d}{dx}(\sin ax) = a\cos x$$

$$\frac{d}{dx}(\cos ax) = -a\sin x$$

$$\int \sin ax \cdot dx = -(\cos ax)/a$$

$$\int \cos ax \cdot dx = \sin ax/a$$