The Open University of Sri Lanka
Department of Electrical and Computer Engineering
Bachelor of Technology (Engineering)
ECX5241– Distributed Parameter Systems
Final Examination - 2011/2012
Duration : Three hours.

Date: 04th March 2012 Time: 0930-1230 hrs

This paper contains three sections A, B & C. Answer **all questions** in section C and **ONE** question each from sections A & B.
Clearly show all the steps and state all the assumptions made.

SECTION A

1) Questions in Section A, are based on the text given on page 4 and 5. Read the paragraph and answer the following questions.

   a. What are the distributed parameters encountered in the scenario described above?
   b. State the assumptions made, when drawing the fluid flow analogy for the traffic flow.
   c. What are the drawbacks of using the above analogy?
   d. Consider an arbitrary length 'Δx' of the highway.
      i. Write an expression for the number of cars within 'Δx'
      ii. Write an equation for the conservation of cars within the section in the time interval 'Δt'.
      iii. Obtain a partial differential equation between ρ & q from above.
   e. Explain in your own words the relationship between ρ and q.

[Total : 50 *marks*]

SECTION B

2) A certain vector field defined in the region $\{-\infty < x < \infty, 0 \le y \le 1\}$ is given by $\underline{A}(x,y) = x^2\underline{i} + 5xy^2\underline{j}$.

   a. Find the divergence of $\underline{A}$. [5 *marks*]
   b. In what region is this field incompressible? Justify. [4 *marks*]
   c. In what region is this field expanding? [5 *marks*]
   d. Find the curl of $\underline{A}$. In what region is this field irrotational? [6 *marks*]
   e. Find the Laplacian of $\underline{A}$. [5 *marks*]

3) Figure Q3 shows a string with finite length L, fixed at both ends. It is released from rest at t = 0, with an initial displacement f(x) at x. The transverse displacement of the string is given by the function u(x,t).

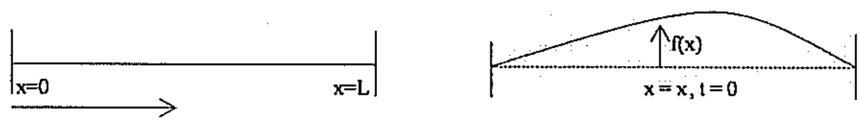

Figure Q3

The motion of the string is governed by the wave equation;

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \quad \text{[Eq3], where } a \text{ is the wave speed.}$$

a. State the boundary conditions for this scenario. [4 *marks*]

b. What are the initial conditions for this problem ? [4 *marks*]

c. Obtain two ordinary differential equations from equation Eq3, by the method of separation of variables, taking the separation constant as μ. [7 *marks*]

d. Find the general solution to Eq3, for $\mu \in \Re$ & $\mu > 0$. [10 *marks*]

SECTION C

4) The Maxwell's equations for electromagnetism in free space can be written

i. $\nabla \circ \underline{B} = 0$ ii. $\nabla \circ \underline{E} = 0$

iii. $\nabla \times \underline{E} + \frac{\partial \underline{B}}{\partial t} = 0$ iv. $\nabla \times \underline{B} - \frac{1}{c}\frac{\partial \underline{E}}{\partial t} = 0$

A vector $\underline{A}$ is defined by $\underline{B} = \nabla \times \underline{A}$, and a scalar ϕ by $\underline{E} = -\nabla\phi - (1/c)(\partial \underline{A}/\partial t)$.
The following condition holds true for $\underline{A}$ and ϕ .

$$\nabla \circ \underline{A} + \frac{1}{c}.\frac{\partial \phi}{\partial t} = 0$$

Show that $\underline{A}$ and ϕ satisfies the wave equations given below.

a. $\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 0$

b. $\nabla^2 \underline{A} - \frac{1}{c^2}\frac{\partial^2 \underline{A}}{\partial t^2} = 0$ [25 *marks*]

5) A steel bar is placed horizontally and is struck with a hammer at t = 0s, as shown in figure Q5a. Let u(x,t) be the displacement of a plane of particles initially at x, at time t. Consider an infinitesimal element of the bar, initially bounded by planes at $x = x_1$ and $x = x_1 + \Delta x$. The element will be temporarily displaced as in figure Q5b.
The bar is elastic, and the force exerted on the element equals the product of its cross sectional area A, strain ε and the modulus of elasticity E, according to Hook's law. Also the force exerted can be expressed using Newton's second law of motion. The mass per unit volume of the bar is ρ.

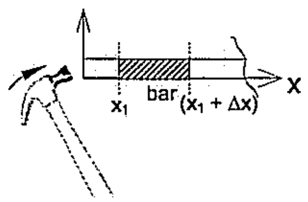

Figure Q5a

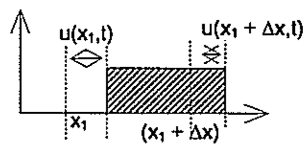

Figure Q5b

a. Derive the governing *P.D.E.* for the longitudinal vibration of the bar.

b. What is the speed of propagation of the wave?

Note :

$$strain, \varepsilon = \frac{elongation\ \ of\ \ element}{unstrained\ \ length\ \ of\ \ element}$$

[25 *marks*]

SECTION A :

Traffic Flow in a Highway

A group of under graduates was given an assignment to do a literature survey on 'mathematical modals of highway traffic' and derive an equation governing the traffic flow based on a simple modal. Given below are some of the extracts of the literature found by them. However some parts of the text are missing and you are requested to assist in deriving the final equation.

Extract A :

One of the mathematical models of traffic flow is the hydrodynamical theory of Lighthill and Whitham (1958). It is a simple theory capable of describing many real-life features of highway traffic with remarkable faithfulness. Consider any section of a straight freeway from $x = a$ to $x = b$, Figure 3. Assume for simplicity that there are no exits or entrances, and all vehicles are on the go. Let the density of cars (number of cars per unit length of highway) at x and t be $\rho(x, t)$, and the flux of cars (number of cars crossing the point x per unit time) be $q(x, t)$.

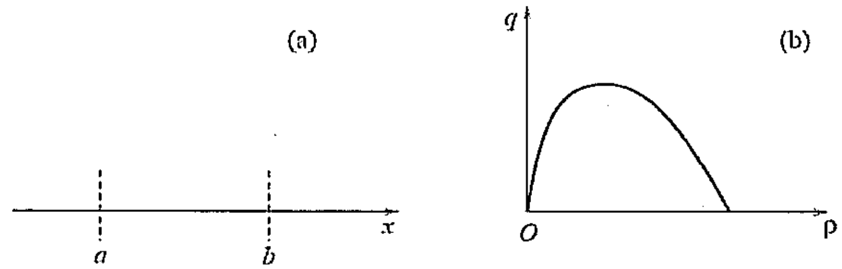

Figure 3: (a). A section of the freeway. (b). The relation between traffic flux rate and traffic density.

Point Noted : The cars within an arbitrary section of the highway are conserved. i.e. The number of cars within the section is equal to the difference between the number of cars entering the section and the number of cars exiting the section, in a given time interval.

Extract B :

Having two unknowns q and ρ, a constitutive relation between ρ and q is needed and must be found by field measurements. Heuristically, q must be zero when there is no car on the road, and zero again when the density attains a maximum (bumper-to-bumper traffic), hence the relation between q and ρ must be nonlinear

$$q = q(\rho) \qquad (3.3)$$

Hence the final governing equation becomes; $\dfrac{\partial \rho}{\partial t} + \left(\dfrac{dq}{d\rho}\right)\dfrac{\partial \rho}{\partial x} = 0.$ EQ - a

Extract C :

3. HYDRODYNAMIC ANALOGIES

Analogies have been often drawn between the flow of fluids and the movement of vehicular traffic. However, the current evidences suggests that the equations developed from hydrodynamic analogies hold good only for high traffic densities and indicates that continuous and steady flow analogies almost totally obscure the fact that each vehicle is individually controlled. Even so, most important traffic-control problems occur only under high-density and other than "free movement" conditions; thus it might appear that better understanding of these analogies is worthwhile.

Principal contributions to this topic have been made by Greenberg, Lighthill and Whitham and Richards. To develop the basic relations, Greenberg's approach will be used. The assumptions are:

1) High-density traffic will behave like a continuous fluid and the corresponding fundamental motion equation for one-dimensional continuous fluid is

$$\frac{dU_s}{dt} = -\frac{c^2}{D}\frac{\partial D}{\partial x} \qquad (22)$$

Where U_s = fluid velocity or space mean speed, mph;
D = density, vehicles/mile;
x = distance, miles,
t = time to move distance x,
c = roadway parameter

Extract A & B :
From lecture notes - *"Wave Propagation, Fall, 2006 MIT, notes by C.C. Mei"*

Extract C :
From -
Traffic Flow Theory Historical Research Perspectives
S.L. DHINGRA[1] & ISHTIYAQ GULL[2]
1.Prof. (Retd.) Transportation Systems Engg & Emeritus Fellow, IIT Bombay-400076
2. M.Tech. Student, Transportation Systems Engg. IIT Bombay-400076
Submitted to: Greenshields Symposium, Woods Hole, MA- July2008

SUMMARY OF VECTOR RELATIONS

	Cartesian Coordinates	*Cylindrical Coordinates*	*Spherical Coordinates*
Coordinate variables	x, y, z	r, ϕ, z	r, θ, ϕ
Vector representation, A =	$\hat{x}A_x + \hat{y}A_y + \hat{z}A_z$	$\hat{r}A_r + \hat{\phi}A_\phi + \hat{z}A_z$	$\hat{r}A_r + \hat{\theta}A_\theta + \hat{\phi}A_\phi$
Magnitude of A, |A|	$\sqrt{A_x^2 + A_y^2 + A_z^2}$	$\sqrt{A_r^2 + A_\phi^2 + A_z^2}$	$\sqrt{A_r^2 + A_\theta^2 + A_\phi^2}$
Position vector from origin to P_1, $\overline{OP_1}$ =	$\hat{x}x_1 + \hat{y}y_1 + \hat{z}z_1$ for $P(x_1, y_1, z_1)$	$\hat{r}r_1 + \hat{z}z_1$ for $P(r_1, \phi_1, z_1)$	$\hat{r}r_1$ for $P(r_1, \theta_1, \phi_1)$
Differential length, dl =	$\hat{x}dx + \hat{y}dy + \hat{z}dz$	$\hat{r}dr + \hat{\phi}rd\phi + \hat{z}dz$	$\hat{r}dr + \hat{\theta}rd\phi + \hat{\phi}r\sin\theta\, d\phi$
Differential surface areas	$ds_x = \hat{x}dydz$ $ds_y = \hat{y}dxdz$ $ds_z = \hat{z}dxdy$	$ds_r = \hat{r}r\,d\phi\,dz$ $ds_\phi = \hat{\phi}drdz$ $ds_z = \hat{z}rdrd\phi$	$ds_r = \hat{r}r^2\sin\theta\, d\theta d\phi$ $ds_\theta = \hat{\theta}r\sin\theta\, drd\phi$ $ds_\phi = \hat{\phi}rdrd\theta$
Differential volume, dv =	$dx\, dy\, dz$	$r\, dr\, d\phi\, dz$	$r^2 \sin\theta\, dr\, d\phi$

A,B,C are vectors and ϕ a scalar function

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})$$

$$\nabla \cdot (\phi\mathbf{A}) = \mathbf{A} \cdot \nabla\phi + \phi\nabla \cdot \mathbf{A}$$

$$\nabla \times (\phi\mathbf{A}) = \nabla\phi \times \mathbf{A} + \phi\nabla \times \mathbf{A}$$

$$\mathbf{A} \times (\nabla \times \mathbf{B}) = \nabla(\mathbf{A} \cdot \mathbf{B}) - (\mathbf{A} \cdot \nabla)\mathbf{B} - (\mathbf{B} \cdot \nabla)\mathbf{A} - \mathbf{B} \times (\nabla \times \mathbf{A})$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B}\cdot)\mathbf{A} - (\mathbf{A} \cdot \nabla\cdot)\mathbf{B}$$

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$$

$$\nabla \times \nabla\phi = 0$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

$$\int_S \mathbf{A} \cdot \hat{\mathbf{n}}dS = \int_v \nabla \cdot \mathbf{A}dv$$ Divergence Theorem

VECTOR OPERATIONS – RECTANGULAR COORDINATES

$$\nabla\alpha = \mathbf{a}_x\frac{\partial\alpha}{\partial x} + \mathbf{a}_y\frac{\partial\alpha}{\partial y} + \mathbf{a}_z\frac{\partial\alpha}{\partial z}$$

$$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\mathbf{A} = \mathbf{a}_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \mathbf{a}_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \mathbf{a}_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla^2\alpha = \frac{\partial^2\alpha}{\partial x^2} + \frac{\partial^2\alpha}{\partial y^2} + \frac{\partial^2\alpha}{\partial z^2} \equiv \nabla\cdot\nabla\alpha$$

$$\nabla^2\mathbf{A} = \mathbf{a}_x\nabla^2 A_x + \mathbf{a}_y\nabla^2 A_y + \mathbf{a}_z\nabla^2 A_z \equiv \nabla(\nabla\cdot\mathbf{A}) - \nabla\times(\nabla\times\mathbf{A})$$

VECTOR OPERATIONS – CYLINDRICAL COORDINATES

$$\nabla\alpha = \mathbf{a}_\rho\frac{\partial\alpha}{\partial\rho} + \mathbf{a}_\phi\frac{1}{\rho}\frac{\partial\alpha}{\partial\phi} + \mathbf{a}_z\frac{\partial\alpha}{\partial z}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\mathbf{A} = \mathbf{a}_\rho\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right) + \mathbf{a}_\phi\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right) + \mathbf{a}_z\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial\phi}\right)$$

$$\nabla^2\alpha = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\alpha}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\alpha}{\partial\phi^2} + \frac{\partial^2\alpha}{\partial z^2}$$

$$\nabla^2\mathbf{A} = \mathbf{a}_\rho\left(\nabla^2 A_\rho - \frac{2}{\rho^2}\frac{\partial A_\phi}{\partial\phi} - \frac{A_\rho}{\rho^2}\right) + \mathbf{a}_\phi\left(\nabla^2 A_\phi + \frac{2}{\rho^2}\frac{\partial A_\rho}{\partial\phi} - \frac{A_\phi}{\rho^2}\right) + \mathbf{a}_z\nabla^2 A_z$$

VECTOR OPERATIONS – SPHERICAL COORDINATES

$$\nabla V = \mathbf{r}^{\sim}\frac{\partial V}{\partial x} + \theta^{\sim}\frac{1}{r}\frac{\partial}{\partial\theta}(V) + \varphi^{\sim}\frac{1}{r\sin(\theta)}\frac{\partial}{\partial\varphi}(V)$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial\theta}(A_\theta\sin(\theta)) + \frac{1}{r\sin(\theta)}\frac{\partial A_\varphi}{\partial\varphi}$$

$$\nabla\times\mathbf{A} = \frac{1}{r^2\sin(\theta)}\begin{vmatrix} \mathbf{r}^{\sim} & r\theta^{\sim} & r\sin(\theta)\varphi^{\sim} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ A_r & rA_\theta & r\sin(\theta)A\varphi \end{vmatrix}$$

$$\nabla\times\mathbf{A} = \frac{\mathbf{r}^{\sim}}{r\sin(\theta)}\left(\frac{\partial}{\partial\theta}A_\varphi\sin(\theta) - \frac{\partial A_\theta}{\partial\theta}\right) + \frac{\theta^{\sim}}{r}\left(\frac{1}{\sin(\theta)}\frac{\partial A_r}{\partial\varphi} - \frac{\partial}{\partial r}(rA_\varphi)\right) + \frac{\varphi^{\sim}}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right)$$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2 V}{\partial\varphi^2}$$