MPZ 3230 – Engineering Mathematics I
Assignment No. 01 – Academic Year 2005

1. (i) Show that the diagonals of a parellogram bisect each other.

   (ii) (a) Show that $\overrightarrow{OA} = \frac{1}{\sqrt{3}}(\underline{i} - \underline{j} + \underline{k})$, $\overrightarrow{OB} = \frac{1}{\sqrt{2}}(\underline{j} + \underline{k})$, $\overrightarrow{OC} = \frac{1}{\sqrt{5}}(-2\underline{i} - \underline{j} + \underline{k})$ are orthogonal to each other.

   (b) Find the unit vector along the direction $\overrightarrow{AC}$.

   (iii) Find the vector which is perpendicular to vector $\underline{a} = \underline{i}$.

   (iv) If $\underline{c}$ is a vector perpendicular to both $\underline{a}$ and $\underline{b}$ then show that $\underline{c}$ is perpendicular to both $(\underline{a} + \underline{b})$ and $(\underline{a} - \underline{b})$.

2. (i) If α and β are scalar constants then prove that $\underline{a} \times (\alpha\underline{b} + \beta\underline{c}) = \alpha\,\underline{a} \times \underline{b} + \beta\,\underline{a} \times \underline{c}$.

   (ii) If $\underline{a} = 2\underline{j} - \underline{k}$, $\underline{b} = \underline{i} - 2\underline{j}$ and $\underline{c} = \underline{i} + \underline{j} + \underline{k}$ then find

   (a) $\underline{a} \times \underline{b}$ (b) $\underline{b} \times \underline{c}$ (c) $(\underline{a} \times \underline{b}).\underline{c}$ (d) $\underline{a} \times (\underline{b} \times \underline{c})$

   (e) $(\underline{a} \times \underline{b}) \times \underline{c}$ (f) $\underline{a}.(\underline{a} \times \underline{b})$ (g) $\underline{a} \times (\underline{a} \times \underline{b})$

   (iii) Prove that the area of a parellogram with sides $\underline{a}$ and $\underline{b}$ is $|\,\underline{a} \times \underline{b}\,|$.

   (iv) Prove that $\frac{d}{dt}(\underline{a} \times \underline{b}) = \underline{a} \times \frac{d\underline{b}}{dt} + \frac{d\underline{a}}{dt} \times \underline{b}$.

3. (i) (a) If the position at time t of a particle is given by, $\underline{r} = (2t^2,\ t^2 - 4t,\ 3t - 5)$ then find the velocity and an acceleration of a given point.

   (b) Find the particle's velocity and an acceleration components to the direction $\underline{i} - 3\underline{j} + 2\underline{k}$.

   (ii) A particle moves so that its position vector is given by $\underline{r} = \cos\omega t\,\underline{i} + \sin\omega t\,\underline{j}$; where ω is a constant show that

   (a) the velocity $\underline{v}$ of the particle is perpendicular to $\underline{r}$.

   (b) the acceleration a is directed forward the origin and has magnitude proportional to the distance from the origin.

   (c) $\underline{r} \times \underline{v}$ = a constant vector.

4. (i) (a) A train of total mass M moves from rest against a resistance which at any time is equal to Mkv, where v is the velocity at this time 't' and k is a constant. Assuming to power P of the engine to be constant, then prove that the equation of motion of the train is

$$M\frac{dv}{dt} = \frac{(P - Mkv^2)}{v}$$

   (b) Show that $Mkv^2 = P(1 - e^{-2kt})$

(c) The power is cut off when the train has speed v_1 and the train is stopped in a further time t_1 by means of a constant breaking force F.
Prove that the new equation of motion of the train is,

$$M\frac{dv}{dt} = -(F + Mkv)$$

and deduce that t_1 is given by, $F = \frac{Mkv_1}{(e^{kt_1} - 1)}$.

(ii) Solve the boundary – value problem.

(a) $\frac{d^2y}{dx^2} = 3x - 2$ initial condition $y(0) = 2, \; y'(1) = -3$.

(iii) Show that the solution of the boundary – value problem
$y''(t) + 4y'(t) + 20y(t) = 16e^{-2t}$; $t \geq 0, \; y(0) = 2, \; y'(0) = 0$ is
$y(t) = e^{-2t}(1 + \sin 4t + \cos 4t)$.

5. Solve the following differential equations.

(i) $\frac{dy}{dx} = e^{x-y} + x^2e^{-y}$

(ii) $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$

(iii) $y\,dy + (1 + y^2)\,dx = 0$

(iv) $\frac{dy}{dx} = \frac{2y - x - 2}{2y - x + 3}$

(v) $\frac{dy}{dx} = \frac{y^4 + x^4}{2yx^3}$

6. (a) Find whether the following differential equations are exact or not and solve differential equations.

(i) $(2xy + 1)\,dx + (x^2 + 4y)\,dy = 0$

(ii) $y\cos x\,dx + 2(\sin x + e^{y^2})\,dy = 0$

(iii) $(y + x^4)\,dx - x\,dy = 0$

(iv) $(x^3 + xy^2 - y)\,dx + x\,dy = 0$

(b) A tank contains 100 *l* of water. A salt solution containing 2 kg of salt per liter flows in at the rate of 3 liters per minute and the well stirred mixture flows out at the same rate.

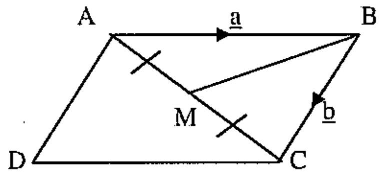
(i) How much salt is in the tank at any time.

(ii) When will the tank have 100 kg of salt ?

Model Answer –MPZ 3230

Assignment # 01 – 2005

(1). (i)



Let $\overrightarrow{AB} = \underline{a}$ and $\overrightarrow{BC} = \underline{b}$

Then $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \underline{a} + \underline{b}$

Let M be a midpoint of $\overrightarrow{AC}$

$\therefore \overrightarrow{AM} = \frac{1}{2}\overrightarrow{AC} = \frac{1}{2}(\underline{a} + \underline{b})$

$\overrightarrow{MB} = \overrightarrow{MA} + \overrightarrow{AB} = -\frac{1}{2}(\underline{a} + \underline{b}) + \underline{a} = \frac{1}{2}(\underline{a} - \underline{b})$

$\overrightarrow{DB} = \overrightarrow{DA} + \overrightarrow{AB} = -\underline{b} + \underline{a} = \underline{a} - \underline{b} \Rightarrow \quad \therefore \overrightarrow{MB} = \frac{1}{2}\overrightarrow{DB}$

$\therefore$ The diagonals of a parellogram bisect each other.

(ii) (a) $\overrightarrow{OA} = \frac{1}{\sqrt{3}}(\underline{i} - \underline{j} + \underline{k}) \quad \overrightarrow{OB} = \frac{1}{\sqrt{2}}(\underline{j} + \underline{k}) \quad \overrightarrow{OC} = \frac{1}{\sqrt{5}}(-2\underline{i} - \underline{j} + \underline{k})$

$\overrightarrow{OA} . \overrightarrow{OB} = \frac{1}{\sqrt{3}}(\underline{i} - \underline{j} + \underline{k}) . \frac{1}{\sqrt{2}}(\underline{j} + \underline{k}) = \frac{1}{\sqrt{6}} . (0 - 1 + 1) = 0$

$\therefore \overrightarrow{OA}$ and $\overrightarrow{OB}$ are orthogonal

$\overrightarrow{OA} . \overrightarrow{OC} = \frac{1}{\sqrt{3}}(\underline{i} - \underline{j} + \underline{k}) . \frac{1}{\sqrt{5}}(-2\underline{i} - \underline{j} + \underline{k}) = \frac{1}{\sqrt{15}}(-2 + 1 + 1) = 0$

$\therefore \overrightarrow{OA}$ and $\overrightarrow{OC}$ are orthogonal

$\overrightarrow{OB} . \overrightarrow{OC} = \frac{1}{\sqrt{2}}(\underline{j} + \underline{k}) . \frac{1}{\sqrt{5}}(-2\underline{i} - \underline{j} + \underline{k}) = \frac{1}{\sqrt{10}}(0 - 1 + 1) = 0$

$\therefore \overrightarrow{OB}$ and $\overrightarrow{OC}$ are orthogonal

(b) $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} = -\frac{1}{\sqrt{3}}(\underline{i} - \underline{j} + \underline{k}) + \frac{1}{\sqrt{5}}(-2\underline{i} - \underline{j} + \underline{k})$

$= \left(-\frac{1}{\sqrt{3}} - \frac{2}{\sqrt{5}}\right)\underline{i} + \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{5}}\right)\underline{j} + \left(-\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}}\right)\underline{k}$

Unit vector along the direction $\overrightarrow{AC} =$

$$\frac{\overrightarrow{AC}}{\left|\overrightarrow{AC}\right|} = \frac{\left[\left(-\frac{1}{\sqrt{3}} - \frac{2}{\sqrt{5}}\right), \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{5}}\right), \left(-\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}}\right)\right]}{\sqrt{\left(-\frac{1}{\sqrt{3}} - \frac{2}{\sqrt{5}}\right)^2 + \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{5}}\right)^2 + \left(-\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}}\right)^2}}$$

$$= \frac{\left[\left(-\frac{1}{\sqrt{3}} - \frac{2}{\sqrt{5}}\right), \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{5}}\right), \left(-\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}}\right)\right]}{\sqrt{9/5}}$$

$$= \frac{\sqrt{5}}{3}\left[\left(-\frac{1}{\sqrt{3}} - \frac{2}{\sqrt{5}}\right), \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{5}}\right), \left(-\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}}\right)\right]$$

(iii) $\underline{a} = \underline{i} = (1,0,0)$

Let $\underline{b} = (a_1, a_2, a_3)$ is a perpendicular vector to vector $\underline{a}$.

Then $\underline{a}.\underline{b} = \underline{0} \Rightarrow \quad \therefore (1,0,0).(a_1, a_2, a_3) = (0,0,0) \Rightarrow a_1 = 0$

$\therefore \underline{b} = (0, a_2, a_3)$; a_2 and a_3 are real numbers.

(iv) Since $\underline{a}$ and $\underline{c}$ are perpendicular vectors,

Then $\underline{a}.\underline{c} = \underline{0}$ ------- (1)

Since $\underline{b}$ and $\underline{c}$ are perpendicular vectors,

Then $\underline{b}.\underline{c} = \underline{0}$ ----- (2)

$(\underline{a}+\underline{b}).\underline{c} = \underline{a}.\underline{c} + \underline{b}.\underline{c} = \underline{0} + \underline{0}$ (by (1) & (2)) $\Rightarrow \therefore (\underline{a}+\underline{b}).\underline{c} = \underline{0}$

$\therefore \underline{c}$ is perpendicular to $(\underline{a}+\underline{b})$

$(\underline{a}-\underline{b}).\underline{c} = \underline{a}.\underline{c} - \underline{b}.\underline{c} = \underline{0} - \underline{0}$ (by (1) and (2)) $\Rightarrow \therefore (\underline{a}-\underline{b}).\underline{c} = \underline{0}$

$\therefore \underline{c}$ is perpendicular to $(\underline{a}-\underline{b})$

(2) (i) Since α and β are scalar constants

Let $\underline{a} = (a_1, a_2, a_3)$, $\underline{b} = (b_1, b_2, b_3)$ and $\underline{c} = (c_1, c_2, c_3)$

$\alpha\underline{b} + \beta\underline{c} = \alpha(b_1, b_2, b_3) + \beta(c_1, c_2, c_3) = [(\alpha b_1 + \beta c_1), (\alpha b_2 + \beta c_2), (\alpha b_3 + \beta c_3)]$

$$\underline{a} \times (\alpha\underline{b} + \beta\underline{c}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ \alpha b_1 + \beta c_1 & \alpha b_2 + \beta c_2 & \alpha b_3 + \beta c_3 \end{vmatrix}$$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ \alpha b_1 & \alpha b_2 & \alpha b_3 \end{vmatrix} + \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ \beta c_1 & \beta c_2 & \beta c_3 \end{vmatrix}$$

$$= \alpha \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} + \beta \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \alpha(\underline{a} \times \underline{b}) + \beta(\underline{a} \times \underline{c})$$

(ii) $\underline{a} = (0,2,-1)$, $\underline{b} = (1,-2,0)$ and $\underline{c} = (1,1,1)$

(a) $$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 2 & -1 \\ 1 & -2 & 0 \end{vmatrix} = (-2,-1,-2)$$

(b) $$\underline{b} \times \underline{c} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -2 & 0 \\ 1 & 1 & 1 \end{vmatrix} = (-2,-1,3)$$

(c) $(\underline{a} \times \underline{b}).\underline{c} = (-2,-1,-2).(1,1,1) = \underline{\underline{-5}}$

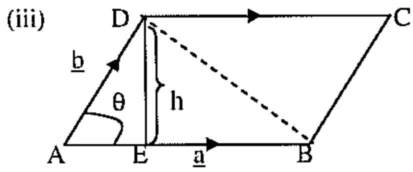
(d) $$\underline{a} \times (\underline{b} \times \underline{c}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 2 & -1 \\ -2 & -1 & 3 \end{vmatrix} = (5,2,4)$$

(e) $$(\underline{a} \times \underline{b}) \times \underline{c} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -2 & -1 & -2 \\ 1 & 1 & 1 \end{vmatrix} = (1,0,0)$$

(f) $\underline{a}.(\underline{a}\times\underline{b})=(0,2,-1).(-2,-1,-2)=0$

(g) $\underline{a}\times(\underline{c}\times\underline{b})=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 0 & 2 & -1\\ -2 & -1 & -2\end{vmatrix}=(-5,2,4)$

(iii)



Let ABCD is a parellogram with $\overrightarrow{AB}=\underline{a},\ \overrightarrow{AD}=\underline{b}$ and

$D\hat{A}B=\theta,\ DE=h$

$\mathrm{Sin}\theta=\dfrac{h}{|\underline{b}|}$

Area of the ABD triangle $=\dfrac{1}{2}.h.|\underline{a}|$

$=\dfrac{1}{2}|\underline{a}|\,\big||\underline{b}|.\sin\theta\big|$

$=\dfrac{1}{2}\big||a|.|b|.\sin\theta\big||\underline{n}|$

Area of the parallelagram $=2.\dfrac{1}{2}\big||\underline{a}||\underline{b}|\sin\theta\big||\underline{n}|$

$=\big||\underline{a}||\underline{b}|\sin\theta\big||\underline{n}|$

$=|\underline{a}\times\underline{b}|$

(iv) Let $\underline{a}=(a_1(t),a_2(t),a_3(t))$

$\underline{b}=(b_1(t),b_2(t),b_3(t))$

Then $\underline{a}\times\underline{b}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}=(a_2b_3-b_2a_3,-a_1b_3+b_1a_3,a_1b_2-a_2b_1)$

$$\frac{d}{dt}(\underline{a}\times\underline{b})=\left(a_2\dot{b}_3+b_3\dot{a}_2-b_2\dot{a}_3-a_3\dot{b}_2\right)\underline{i}+\left(-a_1\dot{b}_3-b_3\dot{a}_1+b_1\dot{a}_3+a_3\dot{b}_1\right)\underline{j}+\left(a_1\dot{b}_2+b_2\dot{a}_1-a_2\dot{b}_1+b_1\dot{a}_2\right)\underline{k}$$

$$=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ \dot{a}_1 & \dot{a}_2 & \dot{a}_3\\ b_1 & b_2 & b_3\end{vmatrix}+\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ a_1 & a_2 & a_3\\ \dot{b}_1 & \dot{b}_2 & \dot{b}_3\end{vmatrix}=\underline{\underline{\frac{d\underline{a}}{dt}\times\underline{b}+\underline{a}\times\frac{d\underline{b}}{dt}}}$$

(3). (i) $\underline{r}=(2t^2,t^2-4t,3t-5)$

(a) Velocity at time $\underline{\underline{t=\underline{v}=\dfrac{d\underline{r}}{dt}=(4t,2t-4,3)}}$

Acceleration at time $t=\underline{f}=\dfrac{d\underline{v}}{dt}=\dfrac{d}{dt}(4t,2t-4,3)$ $\underline{\underline{=(4,2,0)}}$

(b) $|\underline{v}|=\sqrt{4t^2+(2t-4)^2+3^2}=\sqrt{20t^2-16t+25}$

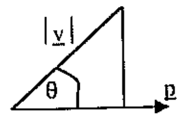
Take $\underline{p}=(1,-3,2)$

$|\underline{P}|=\sqrt{1^2+(-3)^2+2^2}=\sqrt{14}$

$\underline{v}.\underline{P}=|\underline{v}||\underline{P}|\cos\theta$ where θ is the angle between $\underline{v}$ and $\underline{P}$

$$(4t, 2t-4, 3).(1,-3,2) = \sqrt{14(20t^2 - 16t + 25)}\cos\theta$$

$$\cos\theta = \frac{-2t+18}{\sqrt{14(20t^2 - 16t + 25)}}$$



$\therefore$ The component of $|\underline{v}|$ in the direction of $\underline{P} = |\underline{v}|\cos\theta = \dfrac{-2t+18}{\sqrt{14}}$

The unit vector in the direction of $\underline{P} = \dfrac{\underline{P}}{|\underline{P}|} = \dfrac{(1,-3,2)}{\sqrt{14}}$

$\therefore$ The component of the velocity in the direction (1, -3, 2)

$$= \left[\frac{-2t+18}{\sqrt{14}}\right]\frac{(1,-3,2)}{\sqrt{14}} = \left(\frac{-2t+18}{14}, \frac{6t-54}{14}, \frac{4t+36}{14}\right)$$

$$= \left[\frac{-t+9}{7}, \frac{3(t-9)}{7}, \frac{2(t+9)}{7}\right]$$

acceleration $\underline{a} = (4,2,0) \Rightarrow |\underline{a}| = \sqrt{16+4} = \sqrt{20}$

$\underline{a}.\underline{P} = |\underline{a}||\underline{P}|\cos\alpha$; α is the angle between $\underline{a}$ and $\underline{P}$

$\therefore (4,2,0)(1,-3,2) = \sqrt{20.14}\cos\alpha \Rightarrow \quad \cos\alpha = -\dfrac{2}{\sqrt{20}\times\sqrt{14}}$

The component of $|\underline{a}|$ in the direction of $\underline{P} = |\underline{a}|\cos\alpha = \sqrt{20}\dfrac{-2}{\sqrt{20}\times\sqrt{14}} = -\dfrac{2}{\sqrt{14}}$

$\therefore$ The component of the acceleration in the direction (1, -3, 2)

$$= -\frac{2}{\sqrt{14}}\frac{(1,-3,2)}{\sqrt{14}} = \left(-\frac{1}{7}, \frac{3}{7}, -\frac{2}{7}\right)$$

(ii) (a) $\underline{r} = (\cos\omega t, \sin\omega t)$

$$\underline{v} = \frac{d\underline{r}}{dt} = (-\omega\sin\omega t, \omega\cos\omega t)$$

$$\underline{r}.\underline{v} = (\cos\omega t, \sin\omega t).(-\omega\sin\omega t, \omega\cos\omega t)$$

$$= -\omega\sin\omega t\cos\omega t + \omega\sin\omega t\cos\omega t = 0$$

$\therefore \underline{v}$ is perpendicular to $\underline{r}$

(b) acceleration $= \underline{a} = \dfrac{dv}{dt} = \left(-\omega^2\cos\omega t - \omega^2\sin\omega t\right) = -\omega^2\underline{r}$

Direction of $\underline{a}$ is opposite to direction of $\underline{r}$

Therefore the acceleration is directed forward the origin.

$|\underline{a}| = \omega^2$ ---- (1)

The distance from the origin is $|\underline{r}| = \sqrt{\cos^2\omega t + \sin^2\omega t} = 1$

since $\underline{a} = -\omega^2 r \Rightarrow |\underline{a}| = \omega^2|\underline{r}|$

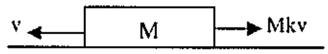
$\therefore |\underline{a}| \; \alpha \; |\underline{r}|$

(c)

$$\underline{r}\times\underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos\omega t & \sin\omega t & 0 \\ -\omega\sin\omega t & \omega\cos\omega t & 0 \end{vmatrix} = \omega\left(\cos^2\omega t + \sin^2\omega t\right)\underline{k} = \omega\underline{k}$$

= Constant vector ($\because \omega$ is a constant)

(4) (i) (a)



Power P; $P=\frac{Fs}{t}= F\frac{s}{t} \Rightarrow P=Fv$

Using Newton's Law, $F=ma$

$$F-Mkv=Ma = Ma+Mkv$$
$$Fv=v(Ma+Mkv)$$
$$P=Mva+Mkv^2 \ (\because P=Fv)$$
$$Mva=P-Mkv^2 \Rightarrow Ma=\frac{P-Mkv^2}{v}$$
$$\underline{\underline{M\frac{dv}{dt}=\frac{P-Mkv^2}{v}}}$$

(b) $M\frac{dv}{dt}=\frac{P-Mkv^2}{v}$

$$\int_0^v\left[\frac{v}{P-Mkv^2}\right]dv=\int_0^t\left[\frac{1}{M}\right]dt \Rightarrow -\frac{1}{2Mk}\left[\ln\left|P-Mkv^2\right|\right]_0^v=\frac{[t]_0^t}{M}$$

$$-\frac{1}{2k}\left[\ln\left|P-Mkv^2\right|-\ln P\right]=t \Rightarrow \ln\left[\frac{P-Mkv^2}{P}\right]=-2kt$$

$$\frac{P-Mkv^2}{P}=e^{-2kt} \Rightarrow \underline{\underline{Mkv^2=P\left(1-e^{-2kt}\right)}}$$

(c) $F=-ma$

$F-Mkv=-ma$

$$F-Mkv=-M\frac{dv}{dt} \Rightarrow \underline{\underline{M\frac{dv}{dt}=-(F+Mkv)}}$$

$$M\frac{dv}{dt}=-(F+Mkv) \Rightarrow -\int_{v_1}^0\frac{dv}{F+Mkv}=\frac{1}{M}\int_0^{t_1}dt$$

$$-\frac{1}{Mk}\left[\ln\left|F+Mkv\right|\right]_{v_1}^0=\frac{1}{M}(t)_0^{t_1} \Rightarrow -\left[\ln F-\ln\left|F+Mkv_1\right|\right]=kt_1$$

$$\ln\left|\frac{F+Mkv_1}{F}\right|=kt_1 \Rightarrow \frac{F+Mkv_1}{F}=e^{kt_1} \Rightarrow \underline{\underline{F=\left[Mkv_1/\left(e^{kt_1}-1\right)\right]}}$$

(ii) (a) $\frac{d^2y}{dx^2}=3x-2$; $y(0)=2$ and $y'(1)=-3$

Integrate with respect to x, $\frac{dy}{dx}=\frac{3x^2}{2}-2x+c_1$

Integrate with respect to x, $y=\frac{3}{2}\frac{x^3}{3}-2\frac{x^2}{2}+c_1x+c_2$

Substitute boundary conditions

$y(0)=2 \Rightarrow 0-0+0+c_2=2 \Rightarrow c_2=2$

$y'(1)=-3 \Rightarrow \frac{3}{2}.1-2.1+c_1=-3 \Rightarrow c_1=-\frac{5}{2}$

$$\underline{\underline{\therefore y(x)=\frac{1}{2}x^3-x^2-\frac{5}{2}x+2}}$$

(iii) $y''(t)+4y'(t)+20y(t)=16e^{-2t} \qquad t \geq 0 \qquad y(0)=2$ and $y'(0)=0$

We have $y(t)=e^{-2t}(1+\sin 4t+\cos 4t)$

$$y'(t)=e^{-2t}(4\cos 4t-4\sin 4t)+(-2)e^{-2t}(1+\sin 4t+\cos 4t)$$
$$=e^{-2t}(2\cos 4t-6\sin 4t-2)$$
$$y''(t)=e^{-2t}(-8\sin 4t-24\cos 4t-4\cos 4t+12\sin 4t+4)$$
$$=e^{-2t}(4\sin 4t-28\cos 4t+4)$$

Then $y''(t)+4y'(t)+20y(t)$

$=e^{-2t}(4\sin 4t-28\cos 4t+4+8\cos 4t-24\sin 4t-8+20+20\sin 4t+20\cos 4t)$

$=e^{-2t}(16)=16e^{-2t}$

Furthermore $y(0)=e^{0}(1+0+1)=2 \Rightarrow y(0)=2$ and $y'(0)=e^{0}(2.1-0-2)=0$

Thus the given relation is a solution to the boundary – value problem.

(5). (i) $\dfrac{dy}{dx}=e^{x-y}+x^2e^{-y}=\dfrac{e^x}{e^y}+\dfrac{x^2}{e^y}$

$\int e^y dy=\int(x^2+e^x)dx \Rightarrow \underline{\underline{e^y=e^x+\frac{x^3}{3}+c}}$, c is constant

(ii) $y-x\dfrac{dy}{dx}=a\left(y^2+\dfrac{dy}{dx}\right)$

$y-(x+a)\dfrac{dy}{dx}=ay^2 \qquad \Rightarrow \qquad (x+a)\dfrac{dy}{dx}=y-ay^2$

$\int\dfrac{dy}{y-ay^2}=\int\dfrac{dx}{x+a} \qquad \Rightarrow \qquad \int\dfrac{dy}{y}+\int\dfrac{dy}{1-ay}=\int\dfrac{dx}{x+a}$

$\ln(y)-\ln(1-ay)=\ln(x+a)+\ln c \qquad \Rightarrow \therefore \underline{\underline{\dfrac{y}{1-ay}=(x+a)c}}$

(iii) $ydy+(1+y^2)dx=0 \quad \Rightarrow \quad \int\dfrac{y}{1+y^2}dy=-\int dx$

$\dfrac{1}{2}\ln(1+y^2)=-x+c \quad \Rightarrow \quad \underline{\underline{\ln(1+y^2)=-2x+c'}}$

(iv) $\dfrac{dy}{dx}=\dfrac{2y-x-2}{2y-x+3} \rightarrow (1)$

Let $v=2y-x$ then $\dfrac{dv}{dx}=2\dfrac{dy}{dx}-1$

Applying (1) ; $\dfrac{1}{2}\left(\dfrac{dv}{dx}+1\right)=\dfrac{v-2}{v+3}$

$\dfrac{dv}{dx}+1=\dfrac{2(v-2)}{v+3} \qquad \Rightarrow \qquad \dfrac{dv}{dx}=\dfrac{v-7}{v+3}$

$\int\dfrac{v+3}{v-7}dv=\int dx \qquad \Rightarrow \qquad \int dv+\int\dfrac{10}{v-7}dv=\int dx$

$v+10\ln|v-7|=x+c \quad \Rightarrow \quad 2y-x+10\ln|2y-x-7|=x+c$

$2(y-x)+10\ln|2y-x-7|=c,$; c is arbitrary constant

(v) $\dfrac{dy}{dx}=\dfrac{y^4+x^4}{2yx^3}$; Let $v={}^{y}\!/_{x}$ $\Rightarrow y=vx \Rightarrow \dfrac{dy}{dx}=v+x\dfrac{dv}{dx}$

$$v+x\frac{dv}{dx}=\frac{v^4x^4+x^4}{2vx^4} \Rightarrow v+x\frac{dv}{dx}=\frac{v^4+1}{2v} \Rightarrow x\frac{dv}{dx}=\frac{v^4+1-2v^2}{2v}$$

$$\int\frac{2v}{\left(v^2-1\right)^2}dv=\int\frac{dx}{x} \Rightarrow -\frac{1}{\left(1-v^2\right)}=\ln x+c \Rightarrow \frac{x^2}{\left(x^2-y^2\right)}=\ln x+c$$

(6). (a) (i) $\underbrace{(2xy+1)}_{g(x,y)}dx+\underbrace{\left(x^2+4y\right)}_{h(x,y)}dy=0$

$\dfrac{\partial g}{\partial y}=2x$ and $\dfrac{\partial h}{\partial x}=2x$

$\therefore$ This equation is exact

$\int(2xy+1)dx+\int 4ydy=c;$ c is an arbitrary constant

$$2y\frac{x^2}{2}+x+4\frac{y^2}{2}=c \Rightarrow x^2y+2y^2+x=c$$

(ii) $\underbrace{y\cos x}_{g(x,y)}dx+\underbrace{2\left(\sin x+e^{y^2}\right)}_{h(x,y)}dy=0$ ---- (1)

$\dfrac{\partial g}{\partial y}=\cos x$ and $\dfrac{\partial h}{\partial x}=2\cos x$

So this equation is not exact.

$$\frac{\dfrac{\partial h}{\partial x}-\dfrac{\partial g}{\partial y}}{g}=\frac{2\cos x-\cos x}{y\cos x}=\frac{1}{y}$$

Integrating factor $=e^{\int\frac{1}{y}dy}=e^{\ln y}=y$

$(1)\times y \Rightarrow \underbrace{y^2\cos x}_{M}dx+\underbrace{2y\left(\sin x+e^{y^2}\right)}_{N}dy=0$ --- (2)

$\dfrac{\partial M}{\partial y}=2y\cos x$ and $\dfrac{\partial N}{\partial x}=2y\cos x$

So equation (2) is exact

$\int y^2\cos dx+\int 2ye^{y^2}dy=c'$; c′ is an arbitrary constant

$y^2\sin x+e^{y^2}=c;$ c is an arbitrary constant

(ii) $\underbrace{\left(y+x^4\right)}_{M}dx\underbrace{-x}_{N}dy=0$ ---- (1)

$\dfrac{\partial M}{\partial x}=1$ and $\dfrac{\partial N}{\partial x}=-1$

So this equation is not exact

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{1-(-1)}{-x}=-2/x$$

Integrating factor $= e\int -2/x\, dx = e^{-2\ln x} = 1/x^2$

$(1)\times 1/x^2 \quad\Rightarrow\quad \frac{1}{x^2}\left(y+x^4\right)dx-\frac{1}{x^2}xdy=0$ --- (2)

$\underbrace{\frac{1}{x^2}\left(y+x^4\right)}_{M}dx-\underbrace{\frac{1}{x}}_{N}dy=0$

$\frac{\partial M}{\partial y}=\frac{1}{x^2} \qquad\qquad \frac{\partial N}{\partial x}=\frac{1}{x^2}$

Then the equation (2) is exact

$\int\left(\frac{y}{x^2}+x^2\right)dx=c'$

$-\frac{y}{x}+\frac{x^3}{3}=c$; c and c′ are arbitrary constants

(iv) $\underbrace{\left(x^3+xy^2-y\right)}_{f(x,y)}dx+\underbrace{xdy}_{g(x,y)}=0$ --- (1)

$\frac{\partial f}{\partial y}=2xy-1 \qquad\qquad \frac{\partial g}{\partial x}=1$

This is not exact

Let $y=vx \;\Rightarrow\; \frac{dy}{dx}=v+x\frac{dv}{dx}\Rightarrow dy=vdx+xdv$ --- (2)

Apply y=vx for equation (1)

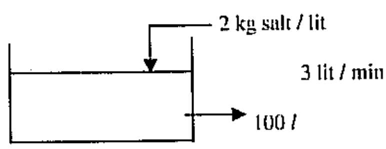
$\left(x^3+xv^2x^2-vx\right)dx+x\left(vdx+xdv\right)=0$

$\left(x^3+xv^2x^2\right)dx+x^2dv=0 \;\Rightarrow\; x^3\left(1+v^2\right)dx+x^2dv=0$

$x\left(1+v^2\right)dx+dv=0 \;\Rightarrow\; \int\frac{dv}{1+v^2}=-\int xdx$

$\tan^{-1}(v)=-\frac{x^2}{2}+c \;\Rightarrow\; \tan^{-1}\left(\frac{y}{x}\right)=-\frac{x^2}{2}+c$

(b) (i)



A(t) = amount of salt in the tank at time t

$\frac{dA(t)}{dt}=6-\frac{3A(t)}{100}$; $A(0)=0$

$\frac{dA(t)}{dt}=\frac{600-3A(t)}{100} \;\Rightarrow\; \int_0^{A(t)}\frac{dA(t)}{600-3A(t)}=\int_0^t\frac{1}{100}dt$

$$-\frac{1}{3}\left[\ln(600-3A(t))\right]_0^{A(t)} = \frac{1}{100}\left[t\right]_0^t$$

$$-\frac{1}{3}\left[\ln 600 - \ln[600-3A(t)]\right] = \frac{t}{100}$$

$$-\frac{1}{3}\frac{\ln 600}{\ln 600 - 3A(t)} = \frac{t}{100} \Rightarrow \qquad -\frac{1}{3}\ln\left[\frac{600}{600-3A(t)}\right] = \frac{t}{100}$$

$$\frac{600}{600-3A(t)} = e^{-3t/100} \qquad \Rightarrow \qquad A(t) = \frac{-600\left(1-e^{-3t/100}\right)}{3e^{-3t/100}}$$

$$A(t) = 200e^{3t/100}\left(1-e^{-3t/100}\right)$$

(ii)

$$t = \frac{100}{3}\ln\left(\frac{600-3A(t)}{600}\right)$$

when $A(t) = 100\text{kg}$

$$t = \frac{100}{3}\ln\left(\frac{600-300}{600}\right) = \frac{100}{3}\ln\left(\frac{1}{2}\right)$$

Engineering Mathematics I – MPZ 3230
Assignment No. 02 – Academic year 2005

Answer all questions

(1) Solve the following differential equations.

(i) $\dfrac{dy}{dx} + y\tan x = \sin 2x$ subject to the condition that $y(x) = 1$ when $x = 0$.

(ii) $\dfrac{dy}{dx} + y\cos x = \dfrac{1}{2}\sin 2x$

(iii) $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = e^{3x}$

(iv) $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} - 4y = \sin 2x$

(v) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - 8y = x^2e^{3x}$

(2) (i) Using the definition of the Laplace Transform, find the Laplace transform of the following functions :

(a) $f(t) = \cos^2(\alpha t)$ (b) $f(t) = (t+1)^2$

(c) $f(t) = e^{-3t}\sin 5t$ (d) $f(t) = t\,e^{-3t}\sin 5t$

(e) $f(t) = \begin{cases} h & 0 < t < T/2 \\ -h & T/2 < t < T \end{cases}$

(ii) Find the inverse Laplace Transforms of the following functions :

(a) $\dfrac{s+2}{s^2+6s+1}$ (b) $\dfrac{2s+3}{s^2+9}$

(iii) Solve the following boundary value problems using the Laplace Transform method :

(a) $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} = 8t$; $y'(0) = y(0) = 0$

(b) $\dfrac{d^2y}{dt^2} - 2\dfrac{dy}{dt} - 3y = 6e^t$; $y(0) = 1$, $y'(0) = 3$

(3) (i) Given $A = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$.

(a) Compute that $A + B$ and $A - C$.
(b) Verify that $A + (B + C) = (A + B) + C$.
(c) Compute AB, BA, and AC^T.

(ii) Express the matrix $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & -1 \\ -3 & 1 & 4 \end{bmatrix}$ as the sum of symmetric and skew symmetric matrix.

(iii) Find the inverse of the matrix $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ by elementary row transformations.

(4) Prove the followings.

(a) $\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

(b) $\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+a & 1 & 1 \\ 1 & 1 & 1+a & 1 \\ 1 & 1 & 1 & 1+a \end{vmatrix} = a^4 + 4a^3.$

(c) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ca & ab \end{vmatrix} = (a-b)(b-c)(c-a)$

(5) (i) For the matrix $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ show that $A^2 - 5A + 7I = 0$ where $I, 0$ are unit and zero matrices. Deduce that $A^{-1} = \frac{1}{7}[5I - A]$. Hence compute A^{-1}.

(ii) Let $A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}$

Find

(a) Determinant of A.
(b) Adjoint of A.
(c) Inverse of A.

Please send your assignment on or before 20.10.2005 to the following address.
Please send your answer with the folder given to you.

Course Coordinator – MPZ 3230
Department of Mathematics & Philosophy of Engineering
Faculty of Engineering Technology
The Open University of Sri Lanka
Nawala – Nugegoda

MPZ 3230 – Assignment No. 02 - Model Answer - Academic Year – 2005

(1). (i). $\frac{dy}{dx} + y \tan x = \sin 2x$ subject to y(x) = 1 when x = 0

$f(x, y) = y \tan x \qquad g(x, y) = 1$

$\frac{\partial f}{\partial y} = \tan x \qquad \frac{\partial g}{\partial x} = 0$

$\frac{\partial f}{\partial y} \neq \frac{\partial g}{\partial x}$ $\therefore$ This is not an exact equation

$$\frac{\left(\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x}\right)}{9} = \frac{\tan x - 0}{1} = \tan x$$

The integrating factor $p(x) = e^{\int \tan x dx} = \frac{1}{\cos x}$

Multiplying by the integrating factor,

$$\frac{1}{\cos x}\frac{dy}{dx} + y\frac{\tan x}{\cos x} = \frac{\sin 2x}{\cos x}$$

$$d\left(\frac{y}{\cos x}\right) = \frac{\sin 2x}{\cos x}$$

$$\frac{y}{\cos x} = 2\int \sin x dx \Rightarrow \frac{y}{\cos x} = -2\cos x + c$$

When x = 0, y(x) = 1 $\Rightarrow c = 3$

$$\therefore \frac{y}{\cos x} = -2\cos x + 3$$

(ii). $\frac{dy}{dx} + y\cos x = \frac{1}{2}\sin 2x$

$f(x, y) = y \cos x \qquad g(x, y) = 1$

$\frac{\partial f}{\partial y} = \cos x \qquad \frac{\partial g}{\partial x} = 0$

$\frac{\partial f}{\partial y} \neq \frac{\partial g}{\partial x}$ $\therefore$ This is not exact equation.

The integrating factor $p(x) = e^{\int \cos x dx} = e^{\sin x}$

Multiplying by the integrating factor,

$$e^{\sin x}\frac{dy}{dx} + y\cos x e^{\sin x} = \frac{1}{2}\sin 2x e^{\sin x}$$

$$d(e^{\sin x} . y) = \frac{1}{2}\sin 2x e^{\sin x}$$

$$e^{\sin x} y = \frac{1}{2}\int \sin 2x e^{\sin x} dx$$

$$e^{\sin x} . y = \sin x e^{\sin x} - e^{\sin x} + c$$

where c is an arbitrary constant

(iii). $\frac{d^2y}{dx^2} - \frac{3dy}{dx} + 2y = e^{3X}$

The characteristic equation is $m^2 - 3m + 2 = 0$

$\therefore\ m_1 = 2\ ,\ m_2 = 2$

The complementary function is, $y = Ae^x + Be^{2x}$;
A & B are arbitrary constants.
Particular integral, $(D^2 - 3D + 2)\, y = e^{3x}$

$$y_p = \frac{1}{(D-2)(D-1)} e^{3x}$$

$$y_p = \frac{1}{D-2} e^{3x} - \frac{1}{D-1} e^{3x} = \frac{e^{3x}}{2}$$

General solution: $y = Ae^x + Be^{2x} + \frac{e^{3x}}{2}$

(iv). $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = \sin 2x$ ⟶ (1)

The characteristic equation is $m^2 - 3m - 4 = 0$

$m_1 = 4$ or $m_2 = -1$

The complementary function is $y_c = Ae^{4x} + Be^{-x}$
Where A & B are arbitrary constants

Particular integral ;
Assume $y_p = a \sin 2x + b \cos 2x$

$y_p' = 2\,(a \cos 2x - b \sin 2x)$

$y_p'' = 4\,(-a \sin 2x - b \cos 2x\,)$

Substitute equation (1) & equating coefficients.

$$\left.\begin{array}{l} -8a + 6b = 1 \\ -8b - 6a = 0 \end{array}\right\} \Rightarrow a = -\frac{2}{25},\ b = \frac{3}{50}$$

$\therefore\ y_p = -\frac{2}{25} \sin 2x + \frac{3}{50} \cos 2x$

General solution : $y = Ae^{4x} + Be^{-x} - \frac{2}{25} \sin 2x + \frac{3}{50} \cos 2x$

(v). $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 8y = x^2e^{3x}$

The characteristic equation is $m^2 + 2m - 8 = 0$

$m_1 = -4$ or $m_2 = 2$

The complementary function is $y_c = Ae^{-4x} + Be^{2x}$
Where A & B are arbitrary constants.
Particular integral; using variation of parameters method,
Complementary functions are $y_1 = e^{2x}$ and $y_2 = e^{-4x}$

$\therefore\ y_1' = 2e^{2x}$ and $y_2' = -4\, e^{-4x}$

The wranstian $w(x) = y_1' y_2 - y_2' y_1 = 6e^{-2x}$

$$\int \frac{y_2 f(x)}{w(x)} dx = \int \frac{x^2 e^x}{6} dx = \frac{1}{6}\left\{e^x x^2 - 2xe^x + 2e^x\right\}$$

$$\therefore A^{-1} = \begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$$

4. (i). $$\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = \begin{vmatrix} a-c & b-a & c-b \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = \begin{vmatrix} a-c & b-a & c-b \\ b+c & c+a & a+b \\ 2c & 2a & 2b \end{vmatrix}$$

$$= 2\begin{vmatrix} a-c & b-a & c-b \\ b+c & c+a & a+b \\ c & a & b \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

(ii). $$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+a & 1 & 1 \\ 1 & 1 & 1+a & 1 \\ 1 & 1 & 1 & 1+a \end{vmatrix} = (4+a)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1+a & 1 & 1 \\ 1 & 1 & 1+a & 1 \\ 1 & 1 & 1 & 1+a \end{vmatrix} = (4+a)\begin{vmatrix} 0 & -a & 0 & 0 \\ 1 & 1+a & 1 & 1 \\ 1 & 1 & 1+a & 1 \\ 1 & 1 & 1 & 1+a \end{vmatrix}$$

$$= (4+a)\, a \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix} = a^3(4+a) = a^4 + 4a^3$$

(c). $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ca & ab \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ c(b-a) & a(c-b) & ab \end{vmatrix} = (a-b)\,(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ -c & -a & ab \end{vmatrix}$$

5. (i). $A = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$ $\qquad A^2 = \begin{pmatrix} 8 & 5 \\ -5 & 3 \end{pmatrix}$

$$A^2 - 5A + 7I = \begin{pmatrix} 8 & 5 \\ -5 & 3 \end{pmatrix} - 5\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix} + 7\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \therefore A^2 - 5A + 7I = 0$$

Multiplying by A^{-1}, then $A - 5I + 7A^{-1} = 0$

$$7A^{-1} = 5I - A$$

$$A^{-1} = \frac{1}{7}(5I - A)$$

By using above equation,

$$A^{-1} = \frac{1}{7}\left[5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}\right] = \begin{pmatrix} \frac{2}{7} & -\frac{1}{7} \\ \frac{1}{7} & \frac{3}{7} \end{pmatrix}$$

(ii). $A = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix}$

(a). Determinant of $A = \begin{vmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{vmatrix} = 54$

$$\text{(b). adjoint of } A = \begin{pmatrix} \begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 4 & 1 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix} & -\begin{vmatrix} 4 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 15 & -3 & -3 \\ -3 & 15 & -3 \\ -3 & -3 & 15 \end{pmatrix}$$

(c). Inverse of $A = A^{-1} = \dfrac{1}{|A|} (\text{adj}(A))^T$

$$\therefore A^{-1} = \frac{1}{54}\begin{pmatrix} 15 & -3 & -3 \\ -3 & 15 & -3 \\ -3 & -3 & 15 \end{pmatrix}$$

$$s^2y(s) - sy(0) - y^1(0) + 2(sy(s) - y(0)) = L\{8t\} = \frac{8}{s^2}$$

Since $y'(0) = y(0) = 0$; Therefore $y(s) = \dfrac{8}{s^3(s+2)}$

$$y(s) = \frac{1}{s} - \frac{2}{s^2} + \frac{4}{s^3} - \frac{1}{s+2}$$

Taking inverse transform $y(t) = 1 - 2t + 2t^2 - e^{-2t}$

(b). $\dfrac{d^2y}{dt^2} - 2\dfrac{dy}{dt} - 3y = 6e^t$; $y(0) = 1$, $y'(0) = 3$

Taking the Laplace transform,

$$s^2y(s) - sy(0) - y'(0) - 2(sy(s) - y(0)) - 3y(s) = L\{6e^t\} = \frac{6}{s-1}$$

Since $y(0) = 1$ & $y'(0) = 3$, $y(s) = \dfrac{s^2+5}{(s-1)(s^2-2s-3)}$

$$\therefore y(s) = -\frac{3}{2} \cdot \frac{1}{(s-1)} + \frac{7}{4} \cdot \frac{1}{(s-3)} + \frac{3}{4} \cdot \frac{1}{(s+1)}$$

Taking inverse transform, $y(t) = -\dfrac{3}{2}e^t + \dfrac{7}{4}e^{3t} + \dfrac{3}{4}e^{-t}$

3. (i). (a). $A + B = \begin{pmatrix} 4 & 1 & -1 \\ 9 & 2 & 7 \\ 3 & -1 & 4 \end{pmatrix}$ $A - C = \begin{pmatrix} -3 & 1 & -5 \\ 5 & -3 & 0 \\ 0 & 1 & -2 \end{pmatrix}$

(b). $A + (B + C) = \begin{pmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{pmatrix} + \begin{pmatrix} 7 & 0 & 4 \\ 4 & 5 & 7 \\ 3 & -2 & 6 \end{pmatrix} = \begin{pmatrix} 8 & 2 & 1 \\ 9 & 5 & 9 \\ 4 & -3 & 7 \end{pmatrix}$ ------------- (1)

$(A + B) + C = \begin{pmatrix} 4 & 1 & -1 \\ 9 & 2 & 7 \\ 3 & -1 & 4 \end{pmatrix} + \begin{pmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{pmatrix} = \begin{pmatrix} 8 & 2 & 1 \\ 9 & 5 & 9 \\ 4 & -3 & 7 \end{pmatrix}$ -------------- (2)

$\therefore (1) \equiv (2) \Rightarrow \therefore A + (B + C) = (A + B) + C$

(c). $AB = \begin{pmatrix} 5 & 3 & 3 \\ 19 & -5 & 16 \\ 1 & -3 & 0 \end{pmatrix}$ $BA = \begin{pmatrix} 0 & 4 & -9 \\ 19 & 3 & -3 \\ 5 & 1 & -3 \end{pmatrix}$

$AC^T = \begin{pmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 4 & 0 & 1 \\ 1 & 3 & -2 \\ 2 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -12 \\ 24 & 4 & 11 \\ 5 & -1 & 6 \end{pmatrix}$

(ii). If A is any square matrix, $A + A^T$ is symmetric and $A - A^T$ is skew-symmetric matrix. We can show that $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$. Where $\frac{1}{2}(A + A^T)$ is symmetric & $\frac{1}{2}(A - A^T)$ is skew – symmetric.

$$\frac{1}{2}(A + A^T) = \frac{1}{2}\begin{pmatrix} 2 & 4 & 1 \\ 4 & 6 & 0 \\ 1 & 0 & 8 \end{pmatrix} \text{ is symmetric matrix}$$

$$\frac{1}{2}(A - A^T) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 7 \\ 0 & 0 & -2 \\ -7 & 2 & 0 \end{pmatrix} \text{ is skew symmetric matrix}$$

Therefore we can write matrix A as the sum of symmetric and skew-symmetric matrix.

(iii). $A = \begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$ we know that A = I. A

$$\begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. A \xrightarrow{R_1 \to \frac{1}{2}R_1}$$

$$\begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. A \xrightarrow{R_2 \to R_2 - 5R_1}$$

$$\begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{5}{2} \\ 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{5}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. A \xrightarrow{R_3 \to R_3 - R_2}$$

$$\begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{5}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{5}{2} & 1 & 0 \\ \frac{5}{2} & -1 & 1 \end{pmatrix}. A \xrightarrow{R_3 \to 2R_3}$$

$$\begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{5}{2} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{5}{2} & 1 & 0 \\ 5 & -2 & 2 \end{pmatrix}. A \quad \begin{matrix} R_1 \longrightarrow R_1 + \frac{1}{2}R_3 \\ R_3 \longrightarrow R_2 - \frac{5}{2}R_3 \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}. A$$

$$\int \frac{y_1 f(x)}{w(x)} dx = \int \frac{x^2 e^{7x}}{6} dx = \frac{e^{7x}}{6.7} - \frac{2xe^{7x}}{6.7.7} + \frac{2e^{7x}}{6.7.7.7}$$

$$\therefore\ y_p(x) = e^{2x}.\frac{1}{6}\left(e^x x^2 - 2xe^x + 2e^x\right) - e^{-4x}\left(\frac{e^{7x}}{6.7} - \frac{2xe^{7x}}{6.7.7} + \frac{2e^{7x}}{6.7.7.7}\right)$$

General solution:

$$Y = Ae^{-4x} + Be^{2x} + e^{3x}\left\{\frac{1}{6}\left(x^2 - 2x + 2\right) - \frac{x^2}{6.7} + \frac{2x}{6.7.7} - \frac{2}{6.7.7.7}\right\}$$

(2). (i). (a). $L\{f(t)\} = \int_0^\alpha e^{-st}\cos^2 \alpha t\, dt$

$$= \int_0^\alpha \cos^2 \alpha t \frac{d}{dt}\left(\frac{e^{-st}}{-s}\right) dt$$

$$= \frac{1}{s} - \frac{\alpha}{s}\int_0^\alpha e^{-st}\sin 2\alpha t$$

$$= \frac{1}{s} - \frac{\alpha}{s}.\frac{2\alpha s}{s^2 + 4\alpha^2} = \frac{2\alpha^2 + s^2}{s(s^2 + 4\alpha^2)}$$

Where $\int_0^\alpha e^{-st}\sin 2\alpha t\, dt = \int_0^\alpha \sin 2\alpha t \frac{d}{dt}\left(\frac{e^{-st}}{-s}\right) dt = \frac{2\alpha s}{s^2 + 4\alpha^2}$

(b). $L\left\{(t+1)^2\right\} = \int_0^\alpha e^{-st}(t+1)^2 dt = \int_0^\alpha e^{-st}\left(t^2 + 2t + 1\right) dt$

$$= \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}$$

(c). $L\{e^{-3t}\sin 5t\} = \int_0^\alpha e^{-st}e^{-3t}\sin 5t\, dt$

$$= \int_0^\alpha \sin 5t \frac{d}{dt}\left(\frac{e^{-(3+s)t}}{-(3+s)}\right) dt$$

$$= \int_0^\alpha \frac{5}{(3+s)}\cos 5t \frac{d}{dt}\left(\frac{e^{-(3+s)t}}{-(3+s)}\right) dt$$

$$= \frac{5}{(3+s)^2} - \frac{25}{(3+s)^2}\int_0^\alpha \sin 5t e^{-(3+s)} dt$$

$$= \frac{5}{(3+s)^2 + 25}$$

(d). $L\{te^{-3t}\sin 5t\} = \int_0^\alpha te^{-(3+s)t}\sin 5t\, dt$

$$= \int_0^\alpha t\sin 5t \frac{d}{dt}\left(\frac{e^{-(3+s)t}}{-(3+s)}\right) dt$$

$$= \left[-\frac{t\sin 5t e^{-(3+s)t}}{(3+s)}\right]_0^\alpha - \int_0^\alpha \frac{e^{-(3+s)t}}{-(3+s)}\frac{d}{dt}(t\sin 5t) dt$$

$$= \frac{1}{(3+s)}\int_0^{\alpha} e^{-(3+s)t}(5t\cos 5t + \sin 5t)dt$$

$$= \frac{5}{3+s}\int_0^{\alpha} e^{-(3+s)t}\, t\cos 5t\, dt + \frac{1}{(3+s)}\int_0^{\alpha} e^{-(3+s)t}\sin 5t\, dt$$

$$= \frac{5}{3+s}\int_0^{\alpha} t\cos 5t \frac{d}{dt}\left(\frac{e^{-(3+s)t}}{-(3+s)}\right)dt + \frac{5}{(3+s)[(s+3)^2+25]}$$

$$L\{te^{-et}\sin 5t\} = \frac{5}{(3+s)}\left[\frac{t\cos 5t.e^{-(3+s)t}}{-(3+s)}\right]_0^{\alpha} - \frac{5}{(3+s)}\int_0^{\alpha}\frac{e^{-(3+s)t}}{(3+s)}\frac{d}{dt}(t\cos 5t)\,dt + \frac{5}{(3+s)[s+3)^2+25]}$$

$$= \frac{5}{(3+s)^2}\int_0^{\alpha} e^{-(3+s)t}(-5t\sin 5t + \cos 5t)dt + \frac{5}{(3+s)[(s+3)^2+25]}$$

$$= -\frac{25}{(s+3)^2}\int_0^{\alpha} e^{-(3+s)t} t\sin 5t\, dt + \frac{5}{(3+s)^2}\left[\frac{1}{(3+s)} - \frac{5}{(3+s)}\int_0^{\alpha} e^{-(3+s)t}\sin 5t\, dt\right]$$

$$+ \frac{5}{(3+s)[(s+3)^2+25]}$$

$$= -\frac{25}{(s+3)^2}\int_0^{\alpha} e^{-(3+s)t}\, t\sin 5t\, dt + \frac{5}{(s+3)^2}\left[\frac{1}{(3+s)} - \frac{25}{(3+s)[(s+3)^2+25]}\right]$$

$$+ \frac{5}{(s+3)[(s+3)^2+25]^2}$$

$$L\{t\, e^{-3t}\sin 5t\} = \frac{10(s+3)}{[(s+3)^2+25]^2}$$

(e). $f(t) = \begin{cases} h; & 0 < t < T/2 \\ -h; & T/2 < t < T \end{cases}$

$$L\{f(t)\} = \int_0^{\alpha} e^{-st} f(t)\,dt = \int_0^{T/2} e^{-st} h\, dt + \int_{T/2}^{T} e^{-st}(-h)dt$$

$$= \frac{h}{s}\left((1 - 2e^{-\frac{sT}{2}} + e^{-ST}\right)$$

(ii). (a). $L^{-1}\left\{\frac{s+2}{s^2+6s+1}\right\} = L^{-1}\left\{\frac{s+3}{(s+3)^2-(\sqrt{8})^2}\right\} - \frac{1}{\sqrt{8}}\, L^{-1}\left\{\frac{\sqrt{8}}{(s+3)^2-(\sqrt{8})^2}\right\}$

$$= e^{-3t}\cosh\sqrt{8t} - \frac{1}{\sqrt{8}}e^{-3t}\sinh\sqrt{8t}$$

(b). $L^{-1}\left\{\frac{2s+3}{s^2+9}\right\} = 2L^{-1}\left\{\frac{s}{s^2+9}\right\} + L^{-1}\left\{\frac{3}{s^2+9}\right\}$

$= 2\cos 3t + \sin 3t$

(iii). (a). $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = 8t$; $y'(0) = y(0) = 0$

Taking the Laplace transform,

MPZ 3230 – Assignment # 03
Academic Year 2005

Answer all questions

(1). (i) Prove that if $v = \ln(x^2 + y^2)$, then $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$

(ii) Show that the equation $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$, is satisfied by,

$$z = \ln\sqrt{(x^2 + y^2)} + \frac{1}{2}\tan^{-1}\left(\frac{y}{x}\right)$$

(iii) In a balanced bridge circuit, $R_1 = R_2R_3/R_4$. If R_2, R_3, R_4, have known tolerances of $\pm x\%, \pm y\%, \pm z\%$ respectively, determine the maximum percentage error in R_1, expressed in term of x, y and z.

(iv) If $z = x^4 + 2x^2y + y^3$ and $x = \gamma\cos\theta$ and $y = \gamma\sin\theta$, find $\dfrac{\partial z}{\partial \gamma}$ *and* $\dfrac{\partial z}{\partial \theta}$ in their simplest forms.

(v) If $z = x\ln(x^2 + y^2) - 2y\tan^{-1}\left(\frac{y}{x}\right)$ verify that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z + 2x$$

(2). (a) Find the equation of the line:

(i) through the point (1, -2, 2) and making the angles of 60°, 120° and 45° with positive direction of axes.

(ii) through the point (1, -3, 4) and perpendicular to the plane $x - 3y + 2z = 4$

(iii) passing through the points (-2, 1, 3) and (4, 2, -2)

(b) Find the coordinates of foot of perpendicular from the point (2, -1, 3) to the plane $3x - 2y - z - 9 = 0$

(c) (i) Find the short distance between the lines

$$L_1 : \frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1} \qquad L_2 : \frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$$

and also

(ii) find the equation of the line of the shortest distance.

(d) Find the equation of the line formed by the intersection of the planes $2x + y - 2z = 5$ and $3x - 6y - 2z = 15$

(e) Find the angle between the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

(3) (i) Use Neton-Raphson method to find a real root of the equation $x^3 - 2x - 5 = 0$ correct to three decimal places.

(ii) Find an approximate value for the definite integral $\int_0^2 \frac{1}{x^2+9} dx$ using

(a) Trapezoidal rule
(b) Simpsons rule

Comparing your answer with the exact value of the integral, obtain an approximate value for the constant π.

(4). (i) The table below given the distances in nautical miles of the visible horizon for the given heights in feet above the earth's surface.

x(height)	100	150	200	250	300
y(distance)	10.63	13.03	15.04	16.81	18.42

Use Newton's forward interpolation to estimate the distance y when x = 130.

(ii) A researcher has obtained following simultaneous equations for his model

$$7x_1 - x_2 = 10$$
$$x_1 + 8x_2 + 2x_3 = -17$$
$$-x_2 + 6x_3 + 2x_4 = 25$$
$$-3x_3 + 7x_4 = 5.$$

Express these in a form suitable for solution either by Jacobi's method or the Gauss-Siedel method, stating clearly which method you have chosen. Illustrate the method you have been, by applying 4 iterations, starting from an initial guess of $(1, -2, 4, 1)^T$.

Please send your assignment **on or before 06.01.2006** to the following address.
Please send your answer with your address (write back of your answer sheet)

Course Coordinator – MPZ 3230
Dept. of Mathematics & Philosophy of Engineering
Faculty of Eng. Technology
The Open University of Sri Lanka
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Nugegoda.

(1). (i). $v = \ln(x^2 + y^2)$

$$\frac{\partial v}{\partial x} = \frac{2x}{x^2 + y^2}, \frac{\partial^2 v}{\partial x^2} = \frac{1}{(x^2 + y^2)^2}\left[2(x^2 + y^2) - 2x \cdot 2x\right]$$

$$= \frac{2(y^2 - x^2)}{(x^2 + y^2)^2}$$

$$\frac{\partial v}{\partial y} = \frac{2y}{x^2 + y^2}, \qquad \frac{\partial^2 v}{\partial y^2} = \frac{1}{(x^2 + y^2)^2}\left[2(x^2 + y^2) - 2y \cdot 2y\right]$$

$$= \frac{2(x^2 - y^2)}{(x^2 + y^2)^2}$$

$$\therefore \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

(ii). $Z = \ln \sqrt{x^2 + y^2} + \frac{1}{2}\tan^{-1}\frac{y}{x}$

$$\frac{\partial z}{\partial x} = \frac{\frac{1}{2}(x^2 + y^2)^{-\frac{1}{2}} \cdot 2x}{\sqrt{x^2 + y^2}} + \frac{1}{2}\left(\frac{(-1)x^{-2}y}{1 + y^2}\right) = \frac{2x - y}{2(x^2 + y^2)}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{2}\left\{\frac{2(x^2 + y^2) - (2x - y) \cdot 2x}{(x^2 + y^2)^2}\right\} = \frac{y^2 - x^2 + xy}{(x^2 + y^2)^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{2} \cdot \frac{2y(x^2 + y^2)^{-1/2}}{\sqrt{x^2 + y^2}} + \frac{1}{2} \cdot \frac{1}{1 + \frac{y^2}{x^2}} \cdot \frac{1}{x} = \frac{2y + x}{2(x^2 + y^2)}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{1}{2}\left\{\frac{(x^2 + y^2) \cdot 2 - (2y + x) \cdot 2y}{(x^2 + y^2)^2}\right\} = \frac{x^2 - y^2 - xy}{(x^2 + y^2)^2}$$

$$\therefore \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

(iii). $R_1 = \frac{R_2 R_3}{R_4}$

Known tolerances are $R_2 = \pm x\%$, $R_3 = \pm y\%$, $R_4 = \pm z\%$

$\therefore R_1 = f(R_2, R_3, R_4)$

Error of the R_1,

$$dR_1 = \frac{\partial R_1}{\partial R_2} \cdot dR_2 + \frac{\partial R_1}{\partial R_3} \cdot dR_3 + \frac{\partial R_1}{\partial R_4} \cdot dR_4$$

$$= \frac{R_3}{R_4} dR_1 + \frac{R_2}{R_4} dR_3 + \frac{(-R_2 R_3)}{R_4^2} dR_4$$

But $dR_1 = \frac{R_3}{R_4} \cdot (\pm x\%) \cdot R_2 + \frac{R_2}{R_4} \cdot (\pm y\%) \cdot R_3 + \frac{(-R_2 R_3)}{R_4^2}(\pm z\%) \cdot R_4$

Because $\dfrac{dR_2}{R_2} = \pm\ x\%,\ \dfrac{dR_3}{R_3} = \pm\ y\%, \dfrac{dR_4}{R_4} = \pm\ z\%$

$$\therefore dR_1 = \left(\frac{R_2 R_3}{R_4}\right)(\pm\ x\%) + \left(\frac{R_2\ R_3}{R_4}\right)(\pm\ y\%) - \left(\frac{R_2 R_3}{R_4}\right)(\pm\ z\%)$$

$dR_1 = R_1\ (\ \pm\ x \pm y \mp z)\ \%$

$$\frac{dR_1}{R_{11}} = \pm(x + y - z)\%$$

Maximum percentage error at $R_1 = |\ \pm (x + y - z)\ \%\ |$

(iv). $z = x^4 + 2\ x^2y + y^3$ and $x = r\cos\theta$ and $y = r\sin\theta$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial r}$$

$$\frac{\partial z}{\partial r} = (4x^3 + 4xy)\cos\theta + (2x^2 + 3y^2)\sin\theta$$

$$\frac{\partial z}{\partial} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial \theta}$$

$$= r\ (4x^3 + 4xy)\ (-\sin\theta) + (2x^2 + 3y^2)\ .\ r\cos\theta$$

$$= (2x^2 + 3y^2)\ x - (4x^3 + 4xy)\ y$$

$$\frac{\partial z}{\partial \theta} = 2x^3 - 4x^3y - xy^2$$

(v). $z = x\ \ln\ (x^2 + y^2) - 2y\ \tan^{-1}\left(\dfrac{y}{x}\right)$

$$\frac{\partial z}{\partial x} = \frac{x.2x}{(x^2 + y^2)} + \ln(x^2 + y^2) - \frac{2y}{1 + \frac{y^2}{x^2}}\ .\ y(-1)x^{-2}$$

$$= 2 + \ln\ (x^2 + y^2)$$

$$\frac{\partial z}{\partial y} = \frac{x.2y}{x^2 + y^2} - 2y\ .\ \frac{1}{1 + \frac{y^2}{x^2}}.\frac{1}{x} + \tan^{-1}\frac{y}{x} = -2\tan^{-1}\frac{y}{x}$$

$$\therefore\quad x\ \frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2x + x\ln(x^2 + y^2) - 2y\tan^{-1}\frac{y}{x}$$

$$\therefore\quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2x + z$$

(2) (i). The equation of the line through the point (1, -2, 2) and making the angles of 60^0, 120^0 and 45^0 with positive direction of axes

$$\frac{x-1}{\cos 60^0} = \frac{y+2}{\cos 120^0} = \frac{z-2}{\cos 45^0}$$

$$\frac{x-1}{1/2} = \frac{y+2}{-1/2} = \frac{z-2}{1/\sqrt{2}}$$

$$\frac{x-1}{1} = \frac{y+2}{-1} = \frac{z-2}{\sqrt{2}}$$

(ii). The equation of the line through the point (1, -3, 4) and perpendicular to the plane x – 3y+2z=4

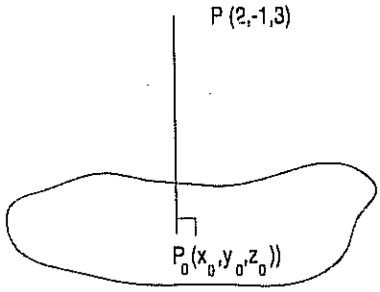
$$\frac{x-1-3}{1}=\frac{y+3}{-3}=\frac{z-4}{2}$$

(iii). The equation of the line passing through the points (-2, 1, 3) and (4, 2, -2)

$$\frac{x+2}{4+2}=\frac{y-1}{2-1}=\frac{z-3}{-2-3}$$

$$\frac{x+2}{6}=\frac{y-1}{1}=\frac{z-3}{-5}$$

(b).



Direction ratios at perpendicular line to given plane is (3, -2, -1) Equation of the perpendicular line to the plane 3x – 2y – z = 9 and through the point p(2, -1, 3)

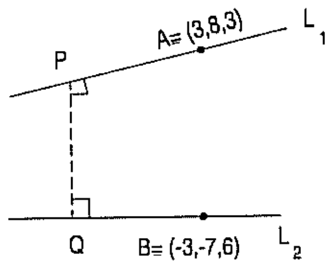
$$\frac{x-2}{3}=\frac{y+1}{-2}=\frac{z-3}{-1}=t$$

General point in the line: (2 + 3t, -1 -2t, 3 –t) p_0 is the foot of perpendicular line,

$\therefore$ $[3x - 2y - z - 9]p_0 = 0$

$3(2 + 3t) -2(-1 -2t) -3 + t = 9 \Rightarrow t = \frac{2}{7}$

$\therefore p_0 \equiv \left(\frac{20}{7}, -\frac{11}{7}, \frac{19}{7}\right)$



(c). (i). $L_1 : \frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$

$L_2 : \frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$

direction ratio of L_1 is $\underline{t}_1 = (3, -1, 1)$ and B ≡ (-3, -7, 6)

direction ratio of L_2 is $\underline{t}_2 = (-3, 2, 4)$

Shortest distance between L_1 & $L_2\} = \frac{\underline{t}_1 \wedge \underline{t}_2}{|\underline{t}_1 \wedge \underline{t}_2|} \cdot \overline{AB}$

$$=\frac{(-6\underline{i}+15\underline{j}+3\underline{k}) \cdot (-6\underline{i}+15\underline{j}+3\underline{k})}{\sqrt{6^2+15^2+3^2}}$$

$$=\sqrt{270}$$

(ii). any point on Line $L_1 = (3t_1 + 3, 8 - t_1, 3 + t_1)$

any point on Line $L_2 = (-3 - 3t_2, 2t_2 - 7, 4t_2 + 6)$

direction ratios of Line PQ $= (3t_1 + 3t_2 + 6, 15 - t_1 - 2t_2, t_1 - 4t_2 - 2)$

Therefore $(3t_1 + 3t_2 + 6) . 3 + (15 - t_1 - 2t_2) . (-1) + 1 . (t_1 - 4t_2 - 3) = 0$

$$11t_1 + 7t_2 = 0 \quad \text{----------} (1)$$

$$(3t_1 + 3t_2 + 6)(-3) + (15 - t_1 - 2t_2)(2) + (t_1 - 4t_2 - 3) . 4 = 0$$

$$7t_1 + 29t_2 = 0 \quad \text{------------------} (2)$$

By (1) & (2) $\quad t_1 = t_2 = 0$

$\therefore$ (3, 8, 3) is a point on the line of the shortest distance.

Direction ratio (6, 15, -3)

Therefore the equation of the line of the shortest distance,

$$\frac{x-3}{6} = \frac{y-8}{15} = \frac{z-3}{-3}$$

(d). $2x + y - 2z = 5$

$3x - 6y - 2z = 15$

The line in intersection is parallel to the vector

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = 14\underline{i} + 2\underline{j} + 15\underline{k}$$

Let z = 0 and solving the equation of two planes, we can obtain a point on the line of intersection,

$$\left.\begin{matrix} 3x - 6y = 15 \\ 2x + y = 5 \end{matrix}\right\} x = 3, y = -1, z = 0$$

Hence (3, -1, 0) is a point on the line of intersection.

Therefore equation of the line of intersection is,

$$\frac{x-3}{14} = \frac{y+1}{2} = \frac{z}{15}$$

(e). $3x - 6y - 2z = 15$, $\quad 2x + y - 2z = 5$

Angle between two planes = θ

$$\cos\theta = \frac{2.3 + (-6).(1) + (-2)(-2)}{\sqrt{3^2 + (-6)^2 + (-2)^2}\sqrt{2^2 + 1^2 + (-2)^2}} = \frac{4}{21}$$

Therefore $\theta = \cos^{-1}\left(\frac{4}{21}\right)$

03. (i) $f(x) = x^3 - 2x - 5$

$f(2) = 2^3 - 2.2 - 5 = -1 < 0$

$f(3) = 3^3 - 2.3 - 5 = 16 > 0$

$\therefore$ According to the intermediate value theorem there exist a root between 2 & 3. By using Newton – Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad , \quad f'(x_n) = 3x^2 - 2$$

when $n = 0$; $x_0 = 2$,

$f(x_0) = f(2) = -1$, $f'(x_0) = 3(2^2) - 2 = 10$

$x_1 = 2 - \frac{(-1)}{10} = 2.1$

$|x_1 - x_0| = |2.1 - 2| = 0.1$

When $n = 1$; $x_1 = 2.1$ $f(x_1) = 2.1^3 - 2 . (2 . 1) - 5 = 0.061$

$f'(x_1) = 3 . (2.1)^2 - 2 = 11.23$

$x_2 = 2.1 - \frac{0.061}{11.23} = 2.095$

$\therefore$ Real root of f (x) = 2.095

(ii). $f(x) = \frac{1}{x^2+9}$

x	0	0.2	0.4	0.6	0.8	1.0	1.2
f(x)	0.1111	0.1106	0.1092	0.1068	0.1037	0.1	0.0958
x	1.4	1.6	1.8	2.0			
f(x)	0.0912	0.0865	0.0817	0.0769			

(a). According to Trapezoidal Rule,

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}[f_0 + f_n = 2(f_1 + f_2 + ... + f_{n-1})]$$

$$\int_0^2 \frac{1}{(x^2+9)}dx = \frac{0.2}{2}\begin{bmatrix}0.1111 + 0.0769 + 2(0.1106 + 0.1092 + 0.1068 + 0.1037 + 0.1 + 0.0958 + 0.0912 \\ + 0.0865 + 0.0817)\end{bmatrix}$$

$= 0.1 \times 1.9590$

$= 0.1959$

(b). According to simpsons rule,

$$\int_a^b f(x)dx = \frac{h}{3} [f_0 + f_n + 4(f_1 + f_3 + ... + f_{n-1}) + 2(f_2 + f_4 + ... + f_{n-2})]$$

$$\int_0^2 \frac{1}{x^2+9}dx = \frac{0.2}{3}\begin{Bmatrix}0.1111 + 0.0769 + 4(0.1106 + 0.1068 + 0.1 + 0.0912 + 0.0817) + 2(0.1092 + 0.1037 \\ + 0.0958 + 0.0865)\end{Bmatrix}$$

$= \frac{0.2}{3} \times 2.9396$

$= 0.1960$

Exact value of $\int_0^2 \frac{1}{x^2+9}$ dx = 0.196

Absolute error in Trapezoidal Method = | 10.196 – 0.195 |

= 0.0001

Absolute error in Simpsons Method = | 10.196 – 0.196 | = 0

Simpsons method is more accurate than Trapezoidal method

$$\int_0^2 \frac{1}{x^2+9}dx = \frac{1}{3}\left[\tan^{-1}\frac{x}{3}\right]_0^2 = \frac{1}{3}\tan^{-1}\frac{2}{3} = \frac{0.1872\pi}{3}$$

But $\int_0^2 \frac{1}{x^2+9} = 0.196$

$\therefore \quad \frac{0.1872\pi}{3} \approx 0.196$

$\therefore \pi \approx 3.1410$

4. (i). Newton's forward interpolation polynomial is,

$$y_s = y_0 + \Delta y_0 s + \Delta^2 y_0 \binom{s}{2} + \ldots\ldots + \Delta^n y_0 \binom{s}{n}$$

where $\binom{s}{n} = \frac{s!}{(s-n)!\,n!}$

$h = 50,\ x_0 = 100,\ \frac{x - x_0}{h} = s \Rightarrow$ since $x = 130$

$s = 0.6$

x (height)	y (distance)	Δ y	Δ^2 y	Δ^3y	Δ^4y
100	10.63				
		2.4			
150	13.03		-0.39		
		2.01		0.15	
200	15.04		-0.24		-0.07
		1.77		0.08	
250	16.81		-0.16		
		1.61			
300	18.42				

We obtain $\quad y_s = 10.63 + 2.4\binom{s}{1} + (-0.39)\binom{s}{2} + 0.15\binom{s}{3} + (-0.07)\binom{s}{4}$

$y_s = 10.63 + 2.4 \times 0.6 + (-0.39)\,(-0.12) + (0.15)\,(0.055) + (-0.07)\,(-0.0336)$

$y_s = 12.12$

When x = 130, estimate distance = 12.12

(ii) $\quad 7x_1 - x_2 = 10$

$x_1 + 8x_2 + 2x_3 = -17$

$-x_2 + 6x_3 + 2x_4 = 25$

$-3x_3 + 7x_4 = 5$

Using Gauss – Siedel method,

$x_1^{(m+1)} = \frac{10}{7} + \frac{x_2^{(m)}}{7}$

$x_2^{(m+1)} = \frac{1}{8}\ (-17 - 2x_3^{(m)} - \frac{10}{7} - \frac{x_2}{7}$

$= -\frac{1}{56}x_2^{(m)} - \frac{1}{4}x_3^{(m)} - \frac{129}{56}$

$x_3^{(m+1)} = \frac{1}{6}\,(25 + x_2^{(m+1)} - 2x_4^{(m)}$

$$= \frac{1}{6}\left(-\frac{1}{56}x_2^{(m)} - \frac{1}{4}x_3^{(m)} - 2x_4^m + \frac{1271}{56}\right)$$

$$x_4^{(m+1)} = \frac{1}{7}\left(5 + \frac{3}{6}\left(-\frac{1}{56}x_2^{(m)} - \frac{1}{4}x_3^{(m)} - 2x_4^{(m)} + \frac{1271}{56}\right)\right)$$

$$= \frac{1}{7}\left(-\frac{1}{112}x_2^{(m)} - \frac{1}{8}x_3^{(m)} - x_4^{(m)} + \frac{1871}{112}\right)$$

m	x_1	x_2	x_3	x_4
0	1.0000	-2.0000	4.0000	1.0000
1	1.1429	-3.2679	3.2887	2.1747
2	0.9617	-3.0674	2.9305	2.0212
3	0.9904	-3.0296	2.9960	2.0493
4	0.9958	-2.9985	2.9838	2.0441

Using Jacobi Method,

$$x_1^{(m+1)} = \frac{1}{7}\{10 + x_2^{(m)}\}$$

$$x_2^{(m+1)} = \frac{1}{8}\ \{-17 - x_1^{(m)} - 2\, x_3^{(m)}\}$$

$$x_3^{(m+1)} = \frac{1}{6}\ \{25 + x_2^{(m)} - 2x_4^{(m)}\}$$

$$x_4^{(m+1)} = \frac{1}{7}\ \{5 + 3x_3^{(m)}\}$$

k	x_1	x_2	x_3	x_4
1	1.0000	-2.0000	4.0000	1.0000
2	1.1428	-3.2500	3.5000	2.4286
3	0.9643	-3.1428	2.8155	2.2142
4	0.9796	-2.9494	2.9048	1.9209
5	1.0072	-2.9736	3.0348	1.9592

MPZ 3230 – Assignment No. 04
Academic Year 2005

Answer all Questions

(1). (i) Find the expansion of cos x

(ii) Find the expansion of $\frac{1}{1+x}$

(iii) Hence show that

$$\frac{\cos x}{1+x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{2} + \frac{13x^4}{24} + ---$$

(2). Show that $\sin^{-1} x = x + \frac{x^3}{6} + \frac{3x^5}{40} + ---$

Assuming the series for e^x, obtain the expansion of $e^x \sin^{-1} x$, upto and including the term in x^4. Hence show that, when x is small the graph of $y = e^x \sin^{-1} x$ approximates to the parabola $y = x^2 + x$.

(3) Evaluate $\int_0^{1/2} \sqrt{1-x^2}dx$

(i) by direct integration
(ii) by expanding as a power series
(iii) by simpson's rule (8 intervals)

(4) The lengths, in millimeters, of 40 bearings were determined with the following results:

16.6	15.3	16.3	14.2	16.7	17.3	18.2	15.6	14.9	17.2
18.7	16.4	19.0	15.8	18.4	15.1	17.0	18.9	18.3	15.9
13.6	18.3	17.2	18.0	15.8	19.3	16.8	17.7	16.8	17.9
17.3	16.6	15.3	16.4	17.3	16.9	14.7	16.2	17.4	15.6

(a) Group the data into six equal width classes between 13.5 and 19.4 mm
(b) Obtain the frequency distribution.
(c) Calculate (i) the mean. (ii) the standard deviation.
(iii) the mode and
(iv) the median of the set of values

(5). A box contains 100 copper plugs, 27 which of which are oversize and 16 undersize. A plug is taken from the box, tested

Case (a) : and replaced: a second plug is then similarly treated.

Case (b) : but not replaced, a second plug is then treated similarly.

Determine the probability for case (a) and (b) that

(i) both plugs are acceptable

(ii) the first is oversize and the second undersize

(iii) one is oversize and the other undersize

(6) The temperature in degrees Celsius in a cool room which is operating properly has the probability distribution function given by,

$$f(x) = kt^2(12-t) \quad 0 < t < 12$$
$$= 0 \quad \text{other wise}$$

(a) Find the value of k

(b) Find the mean and variance of the distribution

(c) Show that the median temperature is about 7.37°C.

(d) The temperature in the cool room is too high if it is over 12°C, find the probability that this occurs.

Please send your assignment **<u>on or before 17.01.2006</u>** to the following address.

<u>Please send your answer with your address (write back of your answer sheet)</u>

Course Coordinator – MPZ 3230
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MPZ 3230 – Assignment No. 04 - Model Answer - Academic Year – 2005

(1). (i). $f(x) = \cos x \quad \Rightarrow \quad f(0) = 1$

$f'(x) = -\sin x \quad \Rightarrow \quad f'(0) = 0$

$f''(x) = -\cos x \quad \Rightarrow \quad f''(0) = -1$

$f'''(x) = \sin x \quad \Rightarrow \quad f'''(0) = 0$

$f''''(x) = \cos x \quad \Rightarrow \quad f''''(0) = 1$

Taylor polynomial for function f(x),

$$f(x) = f(0) + f'(0)\,x + \frac{1}{2!} f''(0)\,x^2 + \ldots\ldots + \frac{1}{n!} f^n(0)\,x^n + \ldots\ldots \text{ for all real x.}$$

$$\text{Therefore } \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots\ldots \text{for all real x.}$$

(ii). $f(x) = \dfrac{1}{1+x} \Rightarrow f(0) = 1$

$$f'(x) = -\frac{1}{(1+x)^2} \Rightarrow f'(o) = -1$$

$$f''(x) = \frac{2}{(1+x)^3} \Rightarrow f''(0) = 2$$

$$f'''(x) = -\frac{6}{(1+x)^4} \Rightarrow f'''(0) = -6$$

By using Taylor polynomial for function f(x),

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots\ldots + (-1)^n x^n + \ldots\ldots \quad \text{for all real x.}$$

(iii). By using part (i) & (ii)

$$\frac{\cos x}{1+x} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots\right)(1 - x + x^2 - x^3 + x^4 + \ldots\ldots) \quad \text{for all real x.}$$

$$= 1 - x + x^2 - x^3 + x^4 + \ldots\ldots - \frac{x^2}{2!} + \frac{x^3}{2!} - \frac{x^4}{2!} + \frac{x^5}{2!} + \frac{x^6}{2!} + \ldots\ldots\ldots + \frac{x^4}{4!} - \frac{x^5}{4!} + \frac{x^6}{4!} + \ldots\ldots\ldots$$

$$= 1 - x + \frac{x^2}{2} - \frac{x^3}{2} + \frac{13x^4}{24} + \ldots\ldots\ldots$$

(2). $f(x) = \sin^{-1} x \qquad f(0) = 0$

$$f'(x) = \frac{1}{\sqrt{1-x^2}} \qquad f'(0) = 1$$

$$f''(x) = \frac{x}{(1-x^2)^{3/2}} \qquad f''(0) = 0$$

$$f'''(x) = \frac{(1+2x^2)}{(1-x^2)^{5/2}} \qquad f'''(0) = 1$$

$$f^{iv}(x) = \frac{3x(3+2x^2)}{(1-x^2)^{7/2}} \qquad f^{iv}(0) = 0$$

$$f^{v}(x) = \frac{3(8x^4+24x^2+3)}{(1-x^2)^{9/2}} \qquad f^{v}(0) = 9$$

$\therefore$ By using Taylor polynomial

$$\sin^{-1}x = 0 + 1\,.\,\frac{x}{1!} + 0 + \frac{x^3}{3!} + 0 + 9\frac{x^5}{5!} + \dots\dots\dots$$

$$= x + \frac{x^3}{6} + \frac{3}{40}x^5 + \dots\dots$$

Series for e^x ; $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} \dots\dots\dots$

$$e^x \sin^{-1} x = (1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} \dots\dots\dots)\left(x + \frac{x^3}{6} + \frac{3}{40}x^5 + \dots\dots\right)$$

$$= x + 0.x^2 + \frac{x^3}{6} + 0.x^4 + \frac{3}{40}x^5 + \dots\dots + x^2 + 0.x^3 + \frac{x^4}{6} + 0.x^5 + \frac{3}{40}x^6 + \dots\dots + x^3 + 0.x^4 + \frac{1}{2.6}x^5 +$$

$$0.x^6 + \frac{3}{2.40}x^7 + \dots\dots + \frac{x^4}{6} + 0.x^5 + \frac{x^6}{6.6}x^6 + \dots\dots + \frac{x^5}{24} + 0.x^6 + \frac{1}{24.6}x^7 + \dots\dots$$

$$e^x \sin^{-1} x = x + x^2 + \frac{4}{6}x^3 + \frac{2}{6}x^4 + \frac{x^5}{5} + \dots\dots$$

When x is small, $x^3 \approx x^4 \approx x^5 \approx 0$ $\therefore$ $e^x \sin^{-1} x = x + x^2$

Therefore $y = e^x \sin^{-1}x$ approximates to parabola $y = x^2 + x$

(3). (i). Direct integration,

$$\int_0^{1/2} \sqrt{1-x^2}\,dx = \int_0^{\pi/6} \cos^2\theta\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/6}(1+\cos 2\theta)d\theta = \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/6} = \left(\frac{\pi}{12} + \frac{\sqrt{3}}{8}\right) = 0.4783$$

Let $x = \sin\theta$

$dx = \cos\theta\, d\theta$

when $x = 0 \Rightarrow \theta = 0$

$x = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$

(ii). Expanding as a power series,

$f(x) = \sqrt{1-x^2}$ $\qquad$ $f(0) = 1$

$f'(x) = -\frac{x}{\sqrt{1-x^2}}$ $\qquad$ $f'(0) = 0$

$f''(x) = -\frac{1}{(1-x^2)^{3/2}}$ $\qquad$ $f''(0) = -1$

$f'''(x) = -\frac{3x}{(1-x^2)^{5/2}}$ $\qquad$ $f'''(0) = 0$

$f^{iv}(x) = \frac{-3+15x^2}{(1-x^2)^{7/2}}$ $\qquad$ $f^{iv}(0) = -3$

By using Maclorin series,

$$f(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \ldots$$

$$\sqrt{1-x^2} = 1 + 0 - \frac{x^2}{2!} + 0 - \frac{3x^4}{4!} + \ldots.$$

$$\sqrt{1-x^2} = 1 - \frac{x^2}{2!} - \frac{3x^4}{4!} + \ldots.$$

$$\int_0^{1/2} \sqrt{1-x^2}\,dx = \int_0^{1/2} \left(1 - \frac{x^2}{2!} - \frac{3x^4}{4!} + \ldots\right) dx$$

$$\int_0^{1/2} \sqrt{1-x^2}\,dx \approx 0.4784$$

(iii). By using Simpson's rule (8 intervals),

$$\int_a^b f(x)\,dx = \frac{h}{3}\ (f_0 + f_n + 4\ (f_1 + f_3 + \ldots\ldots + f_{n-1}) + 2\ (f_2 + f_4 + \ldots.. + f_{n-2}))$$

$$h = \frac{b-a}{n} \quad \therefore h = \frac{\frac{1}{2} - 0}{8} = \frac{1}{16}$$

x	0	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{5}{16}$	$\frac{6}{16}$
x	0	0.0625	0.125	0.1875	0.25	0.3125	0.375
f(x)	1	0.9980	0.9922	0.9823	0.9682	0.9499	0.9270

x	$\frac{7}{10}$	$\frac{8}{16}$
x	0.4375	0.5
f(x)	0.8992	0.8660

$$\int_0^{0.5} \sqrt{1-x^2}\,dx = \frac{0.0625}{3}\{1 + 0.8660 + 4(0.9980 + 0.9823 + 0.9499 + 0.8992) + 2\ (0.9922 + 0.9682 + 0.9270)\}$$

$$= \frac{0.0625}{3}\{1.8660 + 15.3176 + 5.7748\}$$

$$= \frac{0.0625}{3} \times 22.9584$$

$$= 0.4783$$

4. (a). Sample size = 40

Range of the data = 19.4 – 13.5 = 5.9

Class width = $\frac{5.9}{6} = 0.98$

We take class width be 0.9

(a), (b)

Class	f_i	m_i	$f_i m_i$	$f_i m_i^2$
13.5-14.4	2	13.95	27.90	389.2050
14.5-15.4	5	14.95	74.75	1117.5125
15.5-16.4	9	15.95	143.55	2289.6225
16.5-17.4	13	16.95	220.35	3734.9325
17.5-18.4	7	17.95	125.65	2255.4175
18.5-19.4	4	18.95	75.80	1436.4100
	40		688.00	11223.1000

Frequency distribution

(c). (i). mean $= \dfrac{\sum_{i=1}^{n} f_i m_i}{\text{n}} = \dfrac{668}{40} = 16.7$

(ii). The standard deviation= $S = \sqrt{\dfrac{\sum_{i=1}^{n} f_i m_i^2}{n} - \left(\bar{x}\right)^2}$

$$= \sqrt{\frac{11223.1}{40} - (16.7)^2} = 1.2990$$

(iii). Mode $= \text{L} + \left(\dfrac{d_1}{d_1 + d_2}\right)\text{w}$

Model class = 16.5 – 17.4
Class width = w = 17.4 – 16.5 = 0.9
$d_1 = 13 - 9 = 4$
$d_2 = 13 - 7 = 6$
$\text{L} = \dfrac{16.5 + 16.4}{2} = 16.45$

mode $= 16.45 + \left(\dfrac{4}{10}\right) \times 0.9 = 16.81$

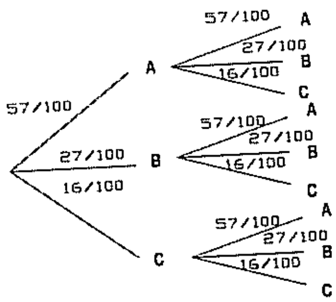
(iv) median $= \text{L} + \left[\dfrac{\frac{n}{2} - F_1}{f}\right] \text{x w}$

f = frequency of median class = 13
w = 17.4 – 16.5 = 0.9
$\text{L} = \dfrac{16.5 + 16.4}{2} = 16.45$
n = 40
F_1 = Cumulative frequency of class before median class = 16

$\therefore$ median $= 16.45 + \left[\dfrac{\frac{40}{2} - 16}{13}\right] 0.9 = 16.45 + \dfrac{4 \times 0.9}{13} = 16.73$

5. Case (a) A plug is taken from the box, tester and replaced: a second plug is then similarly treated

B- No. of Plugs being oversize.
C - No. of Plugs being undersize.
A- No. of Plugs being correct size(Perfect).

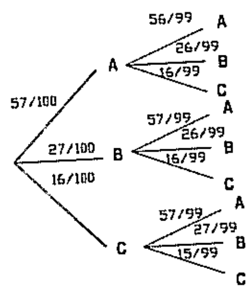


(i). Probability of both plugs being acceptable $\} = \frac{57}{100} \times \frac{57}{100} = 0.3249$

(ii). Probability of the first plug being oversize and the second one is undersize $\} = \frac{27}{100} \times \frac{16}{100} = 0.0432$

(iii). Probability of one plug being oversize and the other one is undersize$\} = \frac{27}{100} \times \frac{16}{100} + \frac{16}{100} \times \frac{27}{100} = 0.0864$

Case (b) A plug is taken from the box, tested but not replace, a second plug is then treated similarly.



(i). Probability of both plugs being acceptable $= \frac{57}{100} \times \frac{56}{99} = 0.3224$

(ii). Probability of the first plug being oversize & the second one is undersize $= \frac{27}{100} \times \frac{16}{99} = 0.0436$

(iii). Probability of one plug being oversize & the other one is undersize = $\frac{27}{100}\times\frac{16}{99}+\frac{16}{100}\times\frac{27}{99}=0.08$

(6). $f(t)=\begin{cases} kt^2(12-t) & ;0<t<12 \\ 0 & ;\text{otherwise} \end{cases}$

(a). $\int_{-\alpha}^{\alpha} f(t)\,dt = 1 \qquad \Rightarrow \therefore \int_0^{12} kt^2(12-t)dt = 1$

$k\int_0^{12}(12t^2-t^3)dt = 1 \qquad \Rightarrow k = \frac{1}{12^3} = \frac{1}{1728}$

(b). Mean = E (t) = $\int_{-\alpha}^{\alpha} t\ f(t)\ dt \qquad \Rightarrow \int_0^{12} t.\ kt^2(12-t)\,dt = 7.2$

$E(t^2) = \int_{-\alpha}^{\alpha} t^2 f(t)\,dt \qquad E(t^2) = \int_0^{12} t^2.\ kt^2(12-t)\,dt = 57.6$

$V(t) = E(t^2) - [E(t)]^2 = 57.6 - (7.2)^2 = 5.76$

Therefore Varíence = 5.76

(c). Let Median be T

Then $\int_{-\alpha}^{T} f(t)\,dt = 0.5$

$\therefore \int_0^{T} kt^2(12-t)dt = 0.5$

Let us assume that median temperature is 7.37^0 c (i.e. $T = 7.37^0$c)

Then $\int_0^{7.37} kt^2(12-t)\,dt = k\left[\frac{t^3.\ 12}{3} - \frac{t^4}{4}\right]_0^{7.37} = 0.4998 \approx 0.5$

Therefore our assumption is correct.
ie. Median Temperature is 7.37^0c.

(d). Let t be temperature in the cool room

$P_r(t > 12) = 1 - p_r(t \le 12)$

$= 1 - \int_0^{12} kt^2(12-t)\,dt = 1 - \underbrace{k\int_0^{12} t^2(12-t)\,dt}_{=1} = 0$

Probability of the temperature in the cool room is too high.