The Open University of Sri Lanka
B.Sc./B.Ed. Degree, Continuing Education Programme
Final Examination -2006/2007
PMU 3291/PME 5291 – Complex Analysis - Paper I
Level 05 - Pure Mathematics

029

Duration :- Two and Half Hours

Date :- 01-12-2006. **Time :- 09.30 a.m. – 12.00 noon.**

Answer Four Questions Only.

01. (i) Solve the following equations and write your answers in the form of $a + ib$ where a and b are real numbers.

(a) $\left(2+\frac{1}{z}\right)^6 = 64$. (b) $\sin z = 7$.

(ii) Is it true that (a) $\left\{(1+i)^2\right\}^{\frac{1}{3}} = \left\{(1+i)^{\frac{1}{3}}\right\}^2$

or (b) $\left\{(1-i)^2\right\}^{\frac{1}{4}} = \left\{(1-i)^{\frac{1}{4}}\right\}^2$?

Justify your answer.

(iii) (a) Prove that the function $f: \mathbb{C} \to [0, \infty)$ given by $f(z) = |z|$ is continuous.

(b) Is the function $g: \mathbb{C} - \{0\} \to (-\pi, \pi]$ given by $g(z) = \operatorname{Arg} z$ continuous? Justify your answer.

02.(i) Let f be the function defined on $\mathbb{R}$ into $\mathbb{R}$ by

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

and let g be the function defined on $\mathbb{C}$ into $\mathbb{C}$ by $g(z) = \begin{cases} z^2 \sin(1/z); & z \neq 0 \\ 0; & z = 0 \end{cases}$

Does the differentiability of f at each point of $\mathbb{R}$ (you don't need to prove this) imply that g is differentiable at each point of $\mathbb{C}$?
Justify your answer.

(ii) Let $G = \{z : |z-2| < 1\} \cup \{z : |z+2| < 1\}$.

Let f be a function defined on G such that $f'(z) = 0$ for each $z \in G$. Does it follow that there exists a constant k such that $f(z) = k$ for each $z \in G$?

Prove your answer.

(iii) Find all the functions defined on $D = \{z : |z| < 1\}$ such that $f(0) = 1 + i$, $|f(z)| = \sqrt{2}$ for each $z \in D$ and f is differentiable on D. Prove your answer.

03. (i) Find the radius of convergence of each of the following power series.

(a) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n+1}-\sqrt{n}}\right)^{n} z^{n}$ (b) $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{n^{n}}\cdot z^{n}$

(ii) Does there exists a power series $\displaystyle\sum_{n=1}^{\infty} a_n (z-3+2i)^n$ which converges at $8-9i$ and diverges at $11+i$? Explain your answer.

(iii) Let f be the function defined on $\mathbb{R}$ into $\mathbb{R}$ by $f(x) = \begin{cases} x^2; & x \geq 0 \\ 0; & x \leq 0 \end{cases}$

which is differentiable at each point of $\mathbb{R}$ (you don't need to prove this).

Does there exist a function $g : \mathbb{C} \to \mathbb{C}$ such that $g(x) = f(x)$ for each $x \in \mathbb{R}$ and g is differentiable at each point of $\mathbb{C}$?

Justify your answer.

04. Evaluate the following integrals, where C_r is the circle $|z| = r$ taken in the counter clockwise direction.

(i) $\displaystyle\int_{C_2} \frac{\cos z}{z^2 - 4z + 3}\, dz$. (ii) $\displaystyle\int_{C_1} \frac{dz}{(z^2+4)(z^2-4)}$.

(iii) $\displaystyle\int_{C_2} \frac{\sin z}{(z-1)^2(z+1)}\, dz$. (iv) $\displaystyle\int_{C_{5/2}} \frac{z^2}{(z+2)^3(z-3)}\, dz$.

05. Expand $f(z)=\dfrac{z^2}{(z+2)(z-3)}$ in a Laurent series valid for

(i) $|z|<2$

(ii) $2<|z|<3$

(iii) $3<|z|$

(iv) $|z-3|<1$.

06. Find and classify the isolated singularities of each of the following functions.

(i) $\dfrac{z^3+8}{z^2(z+2)}$ (ii) $\dfrac{\sin z}{z^2+4}$ (iii) $(z-1)^2 e^{\frac{1}{z-1}}$ (iv) $\dfrac{z}{e^z+1}$.