THE OPEN UNIVERSITY OF SRI LANKA
B.Sc/B.Ed Degree Programme, Continuing Education Programme
APPLIED MATHEMATICS - LEVEL 05
AMU3187/AME 5187 – MATHEMATICAL METHODS II
FINAL EXAMINATION 2006/2007

DURATION: TWO AND HALF-HOURS

| DATE: 21 – 11 – 2006 | TIME: 1.30pm - 4.00pm |
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ANSWER **FOUR** QUESTIONS ONLY.

1. (i) Let $f(x)$ be a function defined in the interval $-\pi < x < \pi$. The Fourier series of $f(x)$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \quad \text{where}$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx,\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx \text{ and } b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

Show that $\frac{1}{\pi}\int_{-\pi}^{\pi} [f(x)]^2 dx = 2a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2$

(ii) Let $f(x) = \cos x, 0 < x < \pi$. Find the Fourier **sine** series of $f(x)$.

(iii) Using parts (i) and (ii) show that

$$\frac{\pi^2}{16} = \frac{2^2}{1^2 \cdot 3^2} + \frac{4^2}{3^2 \cdot 5^2} + \frac{6^2}{5^2 \cdot 7^2} + \cdots.$$

2. Consider the boundary value problem

$$\frac{d^2y}{dx^2} + \mu y = 0 \text{ subject to}$$

$$y(0) - y'(0) = 0 \quad \text{and}$$

$$y(\pi) - y'(\pi) = 0.$$

(i) Show that this is a Sturm-Liouville problem.

(ii) Find the eigenvalues and eigenfunctions of the problem.

(iii) Verify that the eigen functions are mutually orthogonal.

(iii) Obtain a set of functions $\phi$, which are orthonormal in the interval $0 \le x \le \pi$.

3. (i) The Fourier series of a periodic function $f(x)$, $-\pi < x < \pi$ with period $2\pi$ can be written as

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \text{ where}$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx,\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx \text{ and } b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

Show that if $f(x)$ is an **odd function** the Fourier series reduces to

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \text{ with } b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x) \sin nx\, dx.$$

(ii) Find the Fourier series of the function defined by
$f(x) = \sin \alpha x, \quad -\pi < x < \pi$
and $f(x + 2\pi) = f(x)$.
Here $\alpha$ is not an integer.

(iii) Using part (ii) show that

$$\sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)}{\left[9(2n+1)^2 - 1\right]} = \frac{\pi}{18\sqrt{3}}$$

4. The second order differential equation $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$ is known as the Legendre equation. Here $n$ is a real number.

(i) Write down the above differential equation in Sturm-Liouville form.

(ii) Suppose that the above differential equation as a solution of the form $y(x) = \sum_{m=0}^{\infty} c_m x^m$. Then show that $c_{s+2} = -\frac{(n-s)(n+s+1)}{(s+2)(s+1)} c_s \quad s = 0, 1, 2, \cdots$

(iii) If it is given that $c_0 = \frac{(-1)^{n/2} n!}{2^n [(n/2)!]^2}$ ,then using part (ii) show that for even values of $n$, $c_n = \frac{(2n)!}{2^n (n!)^2}$.

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5. (i) The Bessel function of the first kind of order $n$, $J_n(x)$ is given by the expansion $J_n(x)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+n+1)}\left(\frac{x}{2}\right)^{2m+n}$. Suppose it is given that $\frac{d}{dx}\left(x^n J_n(x)\right)=x^n J_{n-1}(x)$ and $\frac{d}{dx}\left(x^{-n} J_n(x)\right)=-x^{-n} J_{n+1}(x)$. Then show that

(a) $\int J_3(x)dx=-J_2(x)-\frac{2}{x}J_1(x)+c$

(b) $\int x[J_0^2(x)]dx=\frac{x^2}{2}\left[J_0^2(x)+J_1^2(x)\right]+c$

(ii) The Gamma function $\Gamma(n)$ is defined as $\Gamma(n)=\int_0^{\infty}x^{n-1}e^{-x}dx$ , $n>0$

(a) Show that $\Gamma(n)=\frac{\Gamma(n+1)}{n}$. Explain how you would use this formula to evaluate $\Gamma(n)$ for negative values of $n$.

(b) Using part (ii) (a) above evaluate $\frac{\Gamma(3)\cdot\Gamma(2.5)}{\Gamma(4.5)}$ and $\Gamma(-3.5)$. You may use $\Gamma(0.5)=\sqrt{\pi}$ without proof.

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6. The Laplacian of $u$ in plane polar coordinates $(r,\theta)$ is given by

$$\nabla^2 u=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

(i) Show that Laplacian of $u$ in Cartesian coordinates is given by

$$\nabla^2 u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}.$$

(ii) Show that If $u(r,\theta)$ satisfies Laplace's equation in polar coordinates, then $v(r,\theta)=u(r^{-1},\theta)$ also satisfies the same equation.

(iii) Find the relation ship between the constants $a$ and $b$ if $u(x,y)=e^{ax}\cos by$ satisfies Laplace's equation in Cartesian coordinates.