The Open University of Sri Lanka
B.Sc./B.Ed. Degree Continuing Education Programme
Final Examination -2006/2007
PMU 2192/PME 4192 - Linear Algebra
Pure Mathematics

115

Duration : Two And Half Hours.

Date : 15-11-2006.	**Time: 09.30 a.m. – 12.00 noon.**

Answer FOUR questions only.

01.(a) Define subspace of a vector space.

(b) Let S be any finite subset of the vector space $\mathbb{R}^3$.
Prove that $<S>$, the span of s, is a subspace of that vector space.

(c) Suppose u and w are two subspaces of a vector space v. Show that $u \cap w$ is also a subspace of v.

(d) Determine which of the following sets are subspaces of $C[a, b]$.

(i) $S = \{f \in C[a, b] : f'(x) = x^2 f(x)\}$

(ii) $T = \{f \in C[a, b] : \int_a^b f(x)dx = 0\}$

(iii) $S = \{f \in C[a, b] : f\dfrac{(a+b)}{2} = 1\}$

where $C[a, b]$ is the set of all continuous functions defined on $[a, b]$.

02. Suppose v is the vector space of all real polynomials of degree at most 3. Let v be the subspace of v consisting of those polynomials of v that vanish at $x = 1$. Let w be the subspace of v consisting of those polynomials of v whose first derivatives vanish at $x = 1$.

(i) Show that $\{x - 1, x^2 - 1, x^3 - 1\}$ is a basis for v

and $\{1, x^2 - 2x, x^3 - 3x\}$ is a basis for w.

(ii) (a) Suppose T is a linear transformation defined by

$T: \mathbb{R}^2 \to \mathbb{R}^3$ and

$$T\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}1\\5\\7\end{pmatrix} \qquad T\begin{pmatrix}5\\6\end{pmatrix}=\begin{pmatrix}2\\3\\4\end{pmatrix}$$

Find $T\begin{pmatrix}13\\18\end{pmatrix}$.

(b) Let $T: v \to v$ be defined by

$T(1) = x^2 + x^4$, $T(x) = x + 1$, $T(x^2) = 1$, $T(x^3) = x^3 + x^2 + 1$, $T(x^4) = x^4$, $T(x^5) = 0$.

Let w be the linear span of $\{1, x^2, x^4\}$.

(i) Show that w is invariant under T.

(ii) Find the matrix of T_w in a suitable basis of w.

03.(a) Define the following terms of a matrix

(i) rank

(ii) nullity

(iii) normal form.

(b) Prove that if there are n homogeneous linear equations in n unknowns, then the system has only the trivial solution if the co-efficient matrix is non-singular.

(c) For the matrix $A = \begin{bmatrix}1 & 2 & 0\\2 & 1 & -6\\2 & -2 & 3\end{bmatrix}$, find P such that $P^{-1}AP$ is a diagonal matrix.

04.(a) Define the following terms:

(i) Characteristic polynomial,

(ii) Characteristic root,

(iii) Characteristic vector.

(b) Prove that if A is an upper or a lower triangular matrix, then the eigenvalues of A are the elements on the diagonal of A.

(c) Find eigenvalues, eigenvectors of the matrix $A = \begin{bmatrix}2 & 2 & -6\\2 & -1 & -3\\2 & -1 & 1\end{bmatrix}$

Show that A is diagonisable.

05.(a) Let A be a square matrix of order n, X and B be column vectors each having n components (or rows).

Prove that the system of equations $AX = B$ possesses a unique solution if the matrix A is non-singular (i.e. A^{-1} exists).

(b) Find all values of a for which the resulting system, given bellow, has

(i) no solution

(ii) a unique solution

(iii) infinitely many solutions.

$$x + 2y - z = 4$$
$$2x - 3y - 5z = 4$$
$$x + 2y + (a^2 - 5)z = a + 2.$$

06.(a) (i) State and prove the Cayley-Hamilton theorem.

(ii) Show that if λ is a characteristic root of an invertible matrix A, then λ^{-1} is a characteristic root of A^{-1}.

(iii) Show that a matrix A and its transpose A' have the same characteristic polynomial.

(b) Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 5 & 4 & 1 \\ 2 & 7 & 2 \end{pmatrix}$.

Evaluate $A^3 - 7A^2 - 9A - 66I$.

Hence find A^{-1}.