The Open University of Sri Lanka
B.Sc./B.Ed. Degree Programme – Level 04
Final Examination – 2006/2007
Pure Mathematics
PMU 2191/PME 4191 – Vector Analysis

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Duration :- Two and a Half Hours

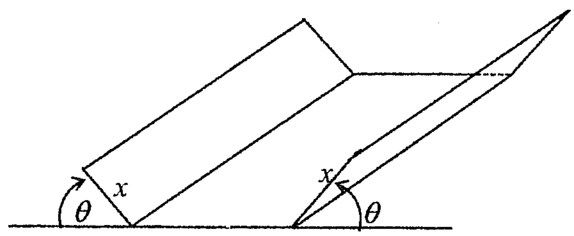
Date :- 11-11-2006. **Time :- 1.00 p.m. – 3.30 p.m.**

Answer Four Questions Only.

01. (a) Define a stationary point of a single valued function $f(x, y)$. Briefly explain how you would determine their nature.

(b) Find and classify the stationary points of the function $f(x, y) = (x + y - 1)(x^2 + y^2)$.

(c) An aqueduct is to be made out of long lead sheets of width 9m. It is constructed by bending each of the sheets into the shape shown below. Find the inclined x of each side and the angle of inclination θ for which the aqueduct's capacity is a maximum.



02. Prove that grad ϕ is a vector normal to the contour curve $\phi(x, y) = \text{constant}$.

(a) The scalar field function ϕ is given by $\phi(x, y) = 5 - x^2 - 2y^2$.

(i) Sketch the contour curves of this function.
(ii) Determine the gradient of ϕ at the point (1, 0) in the direction of the unit vector $\underline{i} + \alpha\underline{j}$
(iii) For what value of α does the directional derivation in part (b) have maximum magnitude?

(b) Let $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$ and $r = \sqrt{x^2 + y^2 + z^2}$. Prove that for any real number n,

(i) $\nabla\left(\frac{1}{r}\right) = \frac{-\underline{r}}{r^3}$

(ii) $\nabla\left(r^2\right) = nr^{n-2}\underline{r}$.

03. (a) Evaluate the surface integral $\iint_D (x^3 y + \cos x)\,dxdy$ where D is the region bounded by $y = x$, $x = \pi/2$ and $y = 0$.

(b) Evaluate $\iint_R e^{x^2+y^2}\,dxdy$ where R is the region bounded by $x^2 + y^2 = 1$, $x^2 + y^2 = 4$, $x = 0$ and $y = 0$.

(c) Find the volume cut out of a sphere of radius a by a circular cone of semi vertical angle $\pi/4$, where vertex is at the centre of the sphere.

(d) The x-coordinate of the centre of gravity of a solid of uniform density ρ is defined by

$\bar{x} = \dfrac{\int_B \rho x dv}{\int_B \rho dv}$. Find the x-coordinate of the centre of gravity of solid of uniform density ρ lying in the region $x \geq 0$, $y \geq 0$, $z \geq 0$, bounded by the three coordinated planes and the sphere $x^2 + y^2 + z^2 = a^2$.

04.(a) State Gauss' Divergence theorem.

(b) Verify the Divergence theorem for the vector field $\underline{F} = \frac{1}{6}xyz\left(x\underline{i} + y\underline{j} + z\underline{k}\right)$ where V is the volume enclosed by the planes $x = 0, x = a, y=0, \; y = b, z = 0, z = c$.

(c) If a region B is bounded by a surface S, show that $\oint_S \frac{\underline{r} \cdot \underline{n}}{r^2}\, dA = \int_B \frac{1}{r^2}\, dV$,

where $\underline{r}$ is the position vector and $r = |\underline{r}|$ and $\underline{n}$ is outward unit normal vector.

(d) Show that $\oint_S \underline{r} \cdot \underline{n}\, dA = 3V$, where $\underline{r}$ is the position vector and S is a closed surface enclosing region B of volume V.

05.(a) Show that a vector field $\underline{F}$ is irrotational if and only if $\int \underline{F} \cdot d\underline{r} = 0$.

(b) Show that if the vector filed $\underline{F}$ is irrotational, then the line integral $\int_C \underline{F} \cdot d\underline{r}$ from a point A to a point B is independent of the path C from A to B.

(c) Show that if $\underline{F}$ is an irrotational vector field, then there exists a scalar field φ such that $\underline{F}$ = grad φ.

(d) Show that the vector field $\underline{F} = yz\underline{i} + (xz + 2yz)\underline{j} + (y^2 + xy)\underline{k}$ is irrotational and find the scalar potential ϕ such that $\underline{F}$ = grad ϕ.

06.(a) If an incompressible fluid has density ρ and velocity $\underline{u}$, then the show that $\frac{\partial \rho}{\partial t} + div(\rho \underline{u}) = 0$.

(b) Show that for an incompressible fluid of uniform constant density, the equation of continuity reduces to $div\; \underline{u} = 0$.

(c) Hence show that the flow specified by the velocity field $\underline{u} = cx\underline{i} + cy\underline{j} - 2cz\underline{k}$, where c is a constant, is possible for an incompressible fluid of uniform constant density.