The Open University of Sri Lanka
B.Sc./B.Ed. Continuing Education Degree Programm
Final Examination -2006/2007
PME 4193/ PMU 2193- Real Analysis

094

Duration : Two and Half Hours

Date: 31/10/2006 Time: 9.30a.m.-12.00noon

Answer Four Questions Only.

01.(a) Show that $\sqrt{3}$ is an irrational number.

(b) Let X and Y be non-empty subsets of $\mathbb{R}$ such that $X \subseteq Y$.
Show that $\inf Y \leq \inf X \leq \sup X \leq \sup Y$.

(c) Find each of the following:

(i) $\bigcap_{n=1}^{\infty}\left(1-\frac{1}{n}, 1+\frac{1}{n}\right)$ (ii) $\sup\left\{x \in \mathbb{Q}: -\sqrt{3} < x < \sqrt{3}\right\}$

(iii) $\lim_{n\to\infty}\left(\frac{\sin n}{n}\right)$ (iv) $\lim_{x\to 2^{+}} x-[x]$

02. (a) Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $\lim_{n\to\infty} x_n = x$. Show that

(i) if $x_n \geq a, \forall n \in \mathbb{N}$, then $x \geq a$.

(ii) if $x_n \leq b, \forall n \in \mathbb{N}$, then $x \leq b$.

(b) Let $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ be sequences of real numbers such that $\lim_{n\to\infty} x_n = l = \lim_{n\to\infty} y_n$ and $x_n \leq z_n \leq y_n$ for $n \in \mathbb{N}$. Show that $\lim_{n\to\infty} z_n = l$. Show that $\lim_{n\to\infty} a^{1/n} = 1$ for $a > 0$.

03. (a) Show that a monotone sequence is convergent if and only if it is bounded.

(b) Let $b > a > 0$. Show that the sequence $\{x_n\}_{n=1}^{\infty}$ given by

$$x_1 = a$$

$$x_{n+1} = \sqrt{\frac{ab^2 + x_n^2}{a+1}} \text{ for } n \in \mathbb{N}$$

is convergent and $\lim_{n\to\infty} x_n = b$.

04. (a) Show that every convergent sequence is Cauchy.

Show that the sequence $\left\{1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}\right\}$ is Cauchy.

(b) Find the limit inferior and limit superior of each of the following sequences:

(i) $\left\{(-1)^n\right\}_{n=1}^{\infty}$ (ii) $\left\{n^2 + (-1)^n n\right\}_{n=1}^{\infty}$

(iii) $\left\{n(\sin n)^2\right\}_{n=1}^{\infty}$ (iv) $\left\{\left(-\frac{1}{n}\right)^n\right\}_{n=1}^{\infty}$

05. (a) In each of the following, give an example that shows that the statements is false.

(i) If $\{x_n\}_{n=1}^{\infty}$ converges and $x_n > M$ for all n, then $\lim_{n\to\infty} x_n > M$.

(ii) Every convergent sequence is monotone.

(iii) The sum of two monotone sequences is monotone.

(b) Test for the convergence of each of the following series:

(i) $\sum_{n=1}^{\infty}\left(\sqrt{n+1} - \sqrt{n}\right)$ (ii) $\sum_{n=1}^{\infty}\left(\sqrt[3]{n^3+1} - n\right)$

(iii) $\sum_{n=1}^{\infty}\frac{1}{3^n + x}, x > 0$ (iv) $\sum_{n=1}^{\infty}\frac{1}{n^n}$

06. (a) Show that $\sum_{n=1}^{\infty} x_n$ is convergent, then $\lim_{n\to\infty} x_n = 0$.

Is the converse true? Justify your answer.

(b) Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers and let $y_n = x_n - x_{n+1}, n \subset \mathbb{N}$.

(i) Prove that the series $\sum_{n=1}^{\infty} y_n$ is convergent if and only if the sequence $\{x_n\}_{n=1}^{\infty}$ is convergent.

(ii) If $\sum_{n=1}^{\infty} y_n$ converges, what is the sum?