THE OPEN UNIVERSITY OF SRI LANKA
B.Sc/B.Ed Degree Programme, Continuing Education Programme
APPLIED MATHEMATICS - LEVEL 05
AMU3189/AME 5189 - STATISTICS II

FINAL EXAMINATION 2006/2007

DURATION: TWO AND HALF-HOURS

| DATE: 13 – 06 – 2007 | TIME: 10.00am – 12.30pm |
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**Statistical Tables are provided. Non-programmable calculators are permitted.**

ANSWER **FOUR** QUESTIONS ONLY.

1. Let $X_1, X_2, \cdots, X_n$ be a random sample from a Pareto distribution with parameters $\alpha$ and $\beta$, so that distribution function is given by

$$F(x;\alpha,\beta) = \begin{cases} 1-\left(\dfrac{\alpha}{x}\right)^{\beta} & x \geq \alpha \\ 0 & \text{otherwise} \end{cases} \qquad \text{where } \alpha>0, \beta>0$$

(a) Find the probability density function, $f(x;\alpha,\beta)$ of $X$.

(b) Write down the likelihood function for the given sample.

(c) Obtain the maximum likelihood estimator (MLE), $\hat{\alpha}$ of $\alpha$ and show that the MLE of $\beta$ is given by $\hat{\beta} = \dfrac{n}{\ln\left(\dfrac{G}{\hat{\alpha}}\right)^n}$.

Here $G$ is the sample geometric mean which is given by $G = (x_1 \times x_2 \times \cdots \times x_n)^{\frac{1}{n}}$.

(d) Let {2, 4, 6, 3, 5, 4, 5, 2, 4, 3} be a random sample from the above distribution. Compute the values of the maximum likelihood estimators of $\alpha$ and $\beta$.

2. Let $X_1, X_2, \cdots, X_n$ be independent and identically distributed random variables from a Bernoulli distribution with parameter $\theta$ so that the density function of $X$ is given by

$$f(x;\theta) = \begin{cases} \theta^x (1-\theta)^{1-x} & x = 0,1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the mean and variance of $X$.

(b) Prove that the density function $f(x,\theta)$ belongs to the exponential family. Hence or otherwise show that $T = \sum_{i=1}^{n} X_i$ is a sufficient statistic for $\theta$.

(c) Using part (b) or otherwise find uniformly minimum variance unbiased estimator (UMVUE) for
(i) expected value of $X$
(ii) variance of $X$

(Hint: If $X_i$'s are independent and identically distributed as Bernoulli ($\theta$) then $\sum_{i=1}^{n} X_i$ is distributed as Binomial ( $n, \theta$)]

3. In order to grow a certain plant the pH value of the soil should be 6.0. A researcher, who is interested in growing these plants in a certain area, examined 16 soil samples and obtained their pH values. The pH value of these 16 soil samples were as follows.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 5.8 | 5.7 | 6.2 | 6.6 | 6.1 | 6.2 | 5.6 | 6.0 |
| 5.9 | 6.0 | 5.7 | 6.1 | 6.3 | 5.8 | 5.9 | 6.3 |

The pH value of the soil in this area is known to be normally distributed with mean $\mu$ and variance $\sigma^2$.

(a) Construct a 95% confidence interval for the mean pH value and interpret your results.

(b) What is the length of the confidence interval you constructed in part (a) above?

(c) Using part (a) or otherwise test the hypothesis that the average pH value of the soil is 6.0. Use a 5% significance level. Clearly state you conclusions.

(d) Construct a 95% confidence interval for the variance of the pH value and interpret your result

4. Let the random variable $X$ denote the claim amounts (Rs.) paid by a motor insurance company. The probability density function of $X$ is given by

$$f(x;\theta) = \frac{1}{6\theta^4} x^3 e^{-\frac{x}{\theta}} \quad ,\theta > 0 \quad x > 0$$

Let $X_1, X_2, \cdots, X_n$ be a random sample of $n$ claim amounts paid by the insurance company.

(a) Show that the moment generating function of $X$ is given by
$M_X(t) = (1-\theta t)^{-4}$.
Hence or otherwise find the mean and the variance of $X$.

(b) Obtain the method of moments estimator for $\theta$.

(c) Is the estimator found in part (b) above a uniformly minimum variance unbiased estimator (UMVUE) for $\theta$ ? Give reasons for your answer.

(d) Suppose {900, 1100, 1600, 1000, 1400, 1200} are six claim amounts paid by the insurance company.

i) Compute the method of moments estimator for $\theta$ based on this sample.

ii) Give an estimate for the variance of the moments estimator computed in part (i) above.

5. A factory produces light bulbs using two processes (say process 1 and process 2). The average lifetime of bulbs produced from process 1 is 150 hrs while for process 2 it is 155hrs. A quality controller selected 16 bulbs from the production output and examined their lifetimes. Let $X_1, X_2, \cdots, X_{16}$ denote the lifetime of these 16 bulbs. From past experience it can be assumed that $X_i \sim N(\mu, 25)$.

The quality controller wishes to test whether or not the selected batch of light bulbs are from process 1.

(a) Explain the following terms in relation to this experiment
i) Null and Alternative Hypothesis
ii) Size of the test
iii) Power of the test

(b) To decide whether the selected light bulbs are from Process 1 or Process 2 a test is carried as follows. The test decides that the light bulbs are from Process 2 if the average lifetime exceeds 152hrs.

i) What is the size of the test?
ii) What is the power of the test?

6. The waiting time, $X$ at a ticket counter, follows a gamma distribution with parameters $q$ and $\theta$ so that the probability density function of $X$ is given by

$$f(x;\theta)=\frac{\theta^{q}\, x^{q-1}\, e^{-\theta x}}{\Gamma(q)} \quad \text{where } x>0, \theta>0 \text{ and } q \text{ is a known constant.}$$

A statistician wishes to test the null hypothesis $H_0:\theta=\theta_0$ against the alternative hypothesis $H_1:\theta=\theta_1$, collected a random sample $X_1, X_2, \cdots, X_n$ from the above distribution.

(a) State whether the hypothesis of interest is simple or composite.

(b) Construct a test of size $\alpha$, using Neyman-Pearson lemma, to test $H_0:\theta=\theta_0$ vs $H_1:\theta=\theta_1$.

(c) Show that if $q=\frac{1}{n}$, the power function $\pi(\theta)$ of the test constructed in part (b) is given by

$$\pi(\theta)=1-(1-\alpha)^{\frac{\theta}{\theta_0}}.$$

(Hint: If $X_i$'s follow a gamma distribution with parameters $q$ and $\theta$ then $\sum_{i=1}^{n} X_i$ has a Gamma distribution with parameters $nq$ and $\theta$.)