The Open University of Sri Lanka
B.Sc. /B.Ed. Degree Programme – Level 05
Final Examination – 2007/2008
Pure Mathematics
PMU 3295 – Ring Theory Paper 2
Duration: - Two and Half Hours.

044

Date: - 07-06-2008 **Time:** - 10.00 a.m. – 12.30 p.m.

Answer Four Questions Only.

(1) (i) Suppose $(R, +, *)$ is a ring and $M(R)$ is the ring of 2×2 matrices with elements in R under usual addition and multiplication. Prove that if $M(R)$ is commutative then $a * b = 0$ for all $a, b \in R$.

(ii) Suppose $(G, +)$ is an abelian group. Define a multiplicative operation $(*)$ on G such that every subgroup of $(G, +)$ is an ideal of $(G, +, *)$.

(2) Suppose R is a ring and A is a right ideal of R. Consider $I(A) = \{x \in R : xa \in A \text{ for all } a \in A\}$. Prove the following,

(i) $I(A)$ is a subring of R.
(ii) $A \subseteq I(A)$.
(iii) A is an ideal of $I(A)$.
(iv) If S is a subring of R such that A is an ideal of S then $S \subseteq I(A)$.

(3) Suppose R is a ring and $a \in R$. Let $I_a = \{r \in R : ra = 0\}$ and $J_a = \{r \in R : ar = 0\}$. Prove that I_a is a left ideal of R and J_a is a right ideal of R. Let R be the set of all 2×2 matrices with integer entries with usual addition and multiplication and let $a = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$. Find I_a and J_a. Is $I_a \cap J_a$ is an ideal of R?.

(4) An ideal I is said to be a minimal ideal of a ring R if $I \neq \{0\}$ and there are no non zero ideals $J\ (\neq I)$ of R such that $J \subset I$.

(i) Prove that I is a minimal ideal of R if and only if the ideal generated by a, $(a) = I$ for all $a \in I$.
(ii) Prove that Z does not have any minimal ideals.
(iii) Find all minimal ideals in Z_{72}. You don't need to justify your answer.

(5) Let R be a commutative ring and let P be a prime ideal of R. Consider the set $S = \{(a, b) : a, b \in R \text{ and } b \notin P\}$. Define the relation $\sim$ on S by $(a, b) \sim (c, d)$ iff $ad - bc \in P$.

(i) Prove that $\sim$ is an equivalence relation on S.

(ii) Consider $S/_\sim = \{[(a, b)] : (a, b) \in S\}$ where $[(a, b)] = \{(c, d) \in S : (a, b) \sim (c, d)\}$.
Define addition and multiplication on $S/_\sim$ by
$[(a, b)] + [(c, d)] = [(a + c, b + d)]$
$[(a, b)] * [(c, d)] = [(ac, bd)]$.
Prove that these operations are well defined.

(iii) Prove that $(S/_\sim, +, *)$ is an integral domain.

(6) Suppose R and S are rings and $\varphi : R \to S$ is a ring homomorphism. State whether following statements are true or false. Justify your answer

(i) If A is a subring of R then $\varphi(A)$ is a subring of S.

(ii) if A is an ideal of R then $\varphi(A)$ is an ideal of S.

(iii) If B is an ideal of S then $\varphi^{-1}(B) = \{a \in R : \varphi(a) \in B\}$ is an ideal of R.