THE OPEN UNIVERSITY OF SRI LANKA
B.Sc/B.Ed Degree Programme, Continuing Education Programme
APPLIED MATHEMATICS - LEVEL 05
AMU3189/AME 5189 - STATISTICS II

FINAL EXAMINATION 2007/2008

094

DURATION: TWO AND HALF-HOURS

| DATE: 21 – 06 – 2008 | TIME: 10.00am – 12.30pm |
|---|---|

**Statistical Tables are provided. Non-programmable calculators are permitted.**

ANSWER **FOUR** QUESTIONS ONLY.

1. Let $X_1, X_2, \cdots, X_n$ be a random sample from a distribution with probability density function

$$f(x;\theta) = \frac{x}{\theta} e^{-\frac{x^2}{2\theta}} \qquad x > 0,\ \theta > 0$$

(a) Show that the $r^{\text{th}}$ moment of $X$ is given by

$$E(X^r) = (2\theta)^{\frac{r}{2}} \Gamma\left(\frac{r}{2} + 1\right)$$

(b) Find an estimator for $\theta$ using
(i) method of moments
(ii) method of maximum likelihood.

(c) Are the estimators found in part (b) above unbiased for $\theta$? Give reasons for you answer.

(d) Using the maximum likelihood estimator for $\theta$ obtained in part b(ii), derive the maximum likelihood estimator for $\sqrt{\theta}$.

(You may use $\Gamma(1.5) = \sqrt{\pi}/2$ and $\pi = 3.14$)

2. Let $X$ be a discrete random variable with the probability distribution

| $x$ | -1 | 1 |
|---|---|---|
| $P(X = x)$ | $\theta$ | $1 - \theta$ |

a) Find the mean and the variance of $X$.

b) Let $x_1, x_2, \cdots, x_5$ be a random sample of 5 values taken from the above distribution. Compute the biases and mean squared errors of each of the following estimators for $\theta$.

i) $2 - x_2$

ii) $\frac{5-(x_1+x_2+\cdots+x_5)}{10}$

iii) $\frac{x_1 - x_2}{2}$

(c) Out of the three estimators, which estimator do you recommend as an estimator for $\theta$? Give reasons for your answer.

3. Let $X_1, X_2, \cdots, X_n$ be a random sample from a normal distribution with mean $\mu$ and variance 1.

(a) Find the Cramer-Rao lower bound for the variance of an unbiased estimator for $\mu^2$.

(b) Using part (a) or otherwise state whether each of the following estimator is a uniformly minimum variance unbiased estimator (UMVUE) for $\mu^2$. Give reasons for your answer.

(i) $\frac{X_1^2+X_2^2+\cdots+X_n^2}{n}$

(ii) $\bar{X}^2 - \frac{1}{n}$

(You may use the result $E(\bar{X}^4) = \mu^4 + \frac{6\mu^2}{n} + \frac{1}{n^2}$ without proof)

4. The actual weight of a '500g' tea packet may be slightly different from 500g. The actual weight of a tea packet is known to be normally distributed with mean $\mu$ and variance 100. A random sample of sixteen '500g' tea packets had the following actual weights (in grams).

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 518 | 507 | 514 | 510 | 485 | 506 | 497 | 502 |
| 488 | 513 | 498 | 506 | 493 | 512 | 510 | 505 |

(a) Construct a 95% confidence interval for the actual mean weight and interpret your result.

(b) What is the length of the confidence interval you constructed in part (a) above?

(c) Using part (a) or otherwise test the hypothesis that the actual mean weight of a randomly chosen tea packet is 500g. Use a 5% significance level. Clearly state your conclusions.

(d) State whether the hypothesis stated in part (c) is simple or composite. Give reasons for your answer.

5. A factory purchases two brands of electric valves that are identical in appearance (say brand *A* and brand *B*). It is known that lifetime (in hours) of electric valves of brand *A* has a normal distribution with mean 1200hrs and standard deviation of 100hrs. The lifetime of electric valves of brand *B* has a normal distribution with mean 1000hrs and standard deviation of 200hrs. The factory has recently purchased a large batch of electric valves on one brand but has lost the information on the brand of these valves. The quality controller wishes to test whether this batch of electric valves are of brand *A*. He proposed to draw a random sample of electric valves from this batch and to carry out a statistical test to decide whether the valves are of brand *A*. He plans to decide that the valves are of brand *A* if the average lifetime of the sample exceeds 1100hrs.

(a) Explain the following terms in relation to this study
   i) Critical Region
   ii) Size of the test
   iii) Power of the test

(b) What is the minimum sample size needed so that the size of the test will be less than 1%?

(c) Find the power of the test based on the sample size obtained in part (b) above.

6. The claim amounts paid by a particular insurance company $X$, follows a gamma distribution with parameters $q$ and $\lambda$ so that the probability density function of $X$ is given by

$$f(x;\lambda) = \frac{\lambda^q \, x^{q-1} \, e^{-\lambda x}}{\Gamma(q)}$$

where $x > 0$, $\lambda > 0$ and $q$ is a known constant.

A researcher collected a random sample $X_1, X_2, \cdots, X_n$ from the above distribution to test the null hypothesis $H_0: \lambda = 2$ against the alternative hypothesis $H_1: \lambda = 4$.

(a) Construct a test of size $\alpha$, using Neyman-Pearson lemma, to test $H_0: \lambda = 2$ vs $H_1: \lambda = 4$.

(b) Show that if $q = \frac{1}{n}$, the power function $\pi(\lambda)$ of the test constructed in part (a) is given by

$$\pi(\lambda) = 1 - \sqrt{(1-\alpha)^{\lambda}}.$$

(Hint: If $X_i's$ follow a Gamma distribution with parameters $q$ and $\lambda$ then $\sum_{i=1}^{n} X_i$ has a Gamma distribution with parameters $nq$ and $\lambda$ .)