THE OPEN UNIVERSITY OF SRI LANKA
B.Sc/B.Ed Degree Programme/Continuing Education Programme
APPLIED MATHEMATICS - LEVEL 05
AMU3187/AME 5187 – MATHEMATICAL METHODS II
FINAL EXAMINATION 2007/2008

DURATION: TWO AND HALF-HOURS

DATE: 19 – 01 – 2008	TIME: 1.00pm - 3.30pm

ANSWER **FOUR** QUESTIONS ONLY

1. Consider the periodic function $f(x)$ defined by
   $f(x) = \sqrt{1-\cos x} \quad -\pi < x < \pi$, and $f(x+2\pi) = f(x)$

   (i) Sketch the graph of $f(x)$ for <u>two</u> periods.

   (ii) Find the Fourier series of $f(x)$.

   (iii) Using part (ii) show that

$$\frac{\pi - 2\sqrt{2}}{4\sqrt{2}} = \frac{1}{3\cdot 5} - \frac{1}{7\cdot 9} + \frac{1}{11\cdot 13} - \frac{1}{15\cdot 17} + \cdots$$

2. Consider the boundary value problem

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \lambda y = 0 \qquad 1 \le x \le 2 \text{ and } y(1) = y(2) = 0$$

   (i) Show that this is a Sturm-Liouville problem.

   (ii) If $z = \ln x$ show that above differential equation can be written as
   $\frac{d^2y}{dz^2} + \lambda y = 0$

   (iii) Using part (ii) or otherwise find the eigenvalues and the eigenfunctions of the above boundary value problem.

3. (i) Let $f_1(x), f_2(x), f_3(x), \cdots$ be a set of real valued functions, which are orthogonal with respect to the weight function $p(x)$ on the interval $a \le x \le b$. If $h_m(x) = \sqrt{p(x)}\, f_m(x)$ $(m = 1,2,3,\cdots)$, then show that $h_1(x), h_2(x), h_3(x), \cdots$ are orthogonal on the interval $a \le x \le b$.

(ii) Let $f_0(x) = a$, $f_1(x) = bx$ and $f_2(x) = c(4x^2 - 1)$ where a, b and c are constants. Show that $f_0(x), f_1(x), f_2(x)$ are orthogonal in the interval $-1 \le x \le 1$ with respect to the weight function $p(x) = \sqrt{1 - x^2}$.

(iii) Find the value of a, b and c so that $f_0(x), f_1(x), f_2(x)$ are orthonormal in the interval $-1 \le x \le 1$ with respect to the weight function $p(x) = \sqrt{1 - x^2}$

4. (i) The n^{th} degree Legendre polynomial is given by

$$P_n(x) = \sum_{m=0}^{M} \frac{(-1)^m (2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^{n-2m}$$

Here $M = \frac{n}{2}$ if n is even and $M = \frac{n-1}{2}$ if n is odd. Using the above equation show that

(a) $P_n(-x) = (-1)^n P_n(x)$

(b) $P_{2n}(0) = \dfrac{(-1)^n (2n)!}{2^{2n} (n!)^2}$

(c) $P_{2n+1}(0) = 0$

(ii) The Legendre polynomial of degree n, $P_n(x)$ is given by the expansion

$(1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x) t^n$. Using this expansion show that

$(n+1) P_{n+1}(x) + n P_{n-1}(x) = (2n+1) x P_n(x), \quad n = 1, 2, 3, \cdots$

Hence show that the norm of $P_n(x)$ is given by $\|P_n(x)\| = \sqrt{\dfrac{2}{2n+1}}$.

5. The Bessel function of the first kind of order n, $J_n(x)$ is given by the expansion
$$e^{\frac{x}{2}(t-t^{-1})} = \sum_{n=-\infty}^{\infty} J_n(x)t^n .$$

(i) Using this expansion show that:

(a) $J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$

(b) $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$

(ii) Using part (i) or otherwise show that:

(a) $J_n''(x) = \frac{1}{4}\left(J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)\right)$

(b) $\frac{d}{dx}\left(J_n^2(x)\right) = \frac{x}{2n}\left(J_{n-1}^2(x) - J_{n+1}^2(x)\right)$

(c) $J_n(x) = \frac{x}{n}\left[J_{n-1} - J_n'(x)\right]$

6. Consider the boundary value problem of Laplacian equation $\nabla^2 u = 0$ in a square $0 < x < \pi$, $0 < y < \pi$ with boundary conditions

$u(x,0) = \sin x(1+\cos x) \qquad 0 < x < \pi$

$u(x,\pi) = x \qquad 0 < x < \pi$

$u(0,y) = 0 \qquad 0 < y < \pi$

$u(\pi,y) = 0 \qquad 0 < y < \pi$

Assuming $u(x,y)$ has a solution of the form $u(x,y) = X(x)\cdot Y(y)$ solve the above boundary value problem.