The Open University of Sri Lanka
B.Sc./B.Ed. Degree Programme
Final Examination – 2007/2008
Applied Mathematics– Level 05
AMU 3182/AME 5182 – Mathematical Methods I

014

Duration :- Two and Half Hours

Date :- 07-01-2008. **Time :- 1.00 p.m. – 3.30 p.m.**

Answer FOUR Questions Only

01. (a) Which of the following equations are separable? For each separable equation, write down the resulting ordinary differential equations for X and Y, where $u(x, y) = X(x)Y(y)$:

(i) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial x \partial y}+\frac{\partial u}{\partial y}=0$.

(ii) $\frac{\partial^2 u}{\partial x^2}+(x+2y)\frac{\partial^2 u}{\partial y^2}=0$.

(iii) $\frac{\partial u}{\partial x}=2\frac{\partial u}{\partial y}+x$.

(b) The eigenvalue problem

$X''(x)+4X'(x)+(4+\lambda)X(x)=0$, where $X(0)=X(1)=0$

has eigenvalues $\lambda_n = n^2\pi^2 (n=1, 2, \cdots)$ and eigenfunctions
$X_n(x)=e^{-2x}\sin n\pi x \;\; (n=1,2,\cdots)$

Use this information and the method of separation of variables to solve the problem described by the equations

$$\frac{\partial^2 u}{\partial x^2}+4\frac{\partial u}{\partial x}+3\frac{\partial u}{\partial y}+4u=0 \quad (0<x<1,\; y>0)$$

$$u(0, y) = u(1, y) = 0 \quad (y \geq 0)$$

$$u(x, 0) = e^{-2x}\sum_{n=1}^{20}\frac{1}{n^2}\sin n\pi x \quad (0<x<1).$$

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02. (a) Find the general solution of the system of simultaneous differential equations given below:

$$\ddot{x}_1 = x_1 + 4x_2$$
$$\ddot{x}_2 = 2x_1 + 3x_2$$

(b) Determine the matrix B such that the three simultaneous differential equations:

$$\dot{y}_1 = -y_1 + 4y_2 - 2y_3$$
$$\dot{y}_2 = -3y_1 + 4y_2$$
$$\dot{y}_3 = -3y_1 + y_2 + 3y_3$$

may be written in the matrix differential form $\dot{y} = By$, where $y = [y_1 \quad y_2 \quad y_3]^T$.

Evaluate B^2 and B^3.

Hence or otherwise, obtain the solution of the given set of equations subject to the conditions $y_1 = 0, \; y_2 = 1, \; y_3 = 2$ at $t = 0$.

03. (a) Find the general solution of each of the following pairs of partial differential equations:

(i) $\dfrac{\partial u}{\partial x} = e^{x+y} + 2x$ and

$\dfrac{\partial u}{\partial y} = e^{x+y} + 2y + \sec^2 y$.

(ii) $\dfrac{\partial u}{\partial x} = \dfrac{2x}{(x^2 - y^2)} + 4x(x - y) + 2(x - y)^2$ and

$\dfrac{\partial u}{\partial y} = \dfrac{-2y}{(x^2 - y^2)} - 4x^2 + 4xy + 3y^2$.

(b) Use the integrating factor method to find the general solution of each of the following partial differential equations:

(i) $\dfrac{\partial u}{\partial y} + \dfrac{4}{y}u = 3y, \quad y \neq 0$

(ii) $\dfrac{\partial u}{\partial x} + u \cot x = \cos x$.

(c) Solve the following eigenvalue problem:

$$2u''(x) + 4u'(x) + \lambda u(x) = 0, \text{ where } u(0) = u(l) = 0.$$

04. (a) (i) Find the general solution of the boundary value problem,

$$3x^2u''(x)+13xu'(x)-8u(x)=0,\ x>0, \text{ where } \lim_{x\to 0}u(x) \text{ is bounded.}$$

(ii) State whether the boundary value problem, $3x^2u''(x)+13xu'(x)-8u(x)=0$, $x>0$, has any solution for which $\lim_{x\to 0}u(x)$ and $\lim_{x\to\infty}u(x)$ are both bounded. Justify your answer.

(b) Solve the following boundary value problems:

(i) $\dfrac{\partial^2 u}{\partial x\partial y}=x^2$, given that when $x=0$, $\dfrac{\partial u}{\partial y}=y^2+2y$ and when $y=1$, $u(x, y)=1$.

(ii) $\dfrac{\partial^2 u}{\partial y^2}=xy^2$, given that $u(x, 0)=x^2-2$ and when $x-y=0$, $\dfrac{\partial u}{\partial y}=x^3$.

05. (a) Find the general solution of the following systems of simultaneous differential equations:

$$\frac{dx_1}{dt}=2x_1+3x_2+2t$$

$$\frac{dx_2}{dt}=2x_1+x_2+2e^{2t}$$

(b) Find a sinusoidal particular solution of the system of equations:

$$4\ddot{x}_1+\ddot{x}_2+3\dot{x}_1+6x_1+4x_2=2\sin 2t-\cos 2t$$

$$\ddot{x}_1+4\ddot{x}_2+3\dot{x}_2+3x_2+4x_1=\cos 2t$$

(c) Solve the differential equation $7x^2\dfrac{d^2y}{dx^2}+2x\dfrac{dy}{dx}=0$.

Find the particular solution for which $y=1$ and $\dfrac{dy}{dx}=1$ when $x=1$.

06. (a) (i) Write down the chain rule for partial derivatives.

(ii) Use the transformation $\zeta = \frac{y}{x}$ and $\phi = xy$ and the chain rule to write $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial y^2}$ in terms of the partial derivatives of u with respect to ζ and ϕ. Hence show that the second order partial differential equation

$$x^2 \frac{\partial^2 u}{\partial x^2} - y^2 \frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 0$$

leads to the equation $\frac{\partial^2 u}{\partial \zeta \, \partial \phi} = 0$ and obtain the general solution for $u(x, y)$. Here, u is a function of the two variables x and y.

(b) Suppose $u(x, y) = F(y^2 - x) + g(y^2 + x)$ is the general solution of a partial differential equation in the region $y > 0$. Find the particular solution which satisfies the additional conditions $u(x, 1) = 1 + x^2$, $\frac{\partial u}{\partial y}(x,1) = 2x$.