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The Open University of Sri Lanka
B.Sc. / B.Ed. Degree Programme – Level 03
Open Book Test (OBT) – 2009/2010
Pure Mathematics
PUU 1141 – Foundation of Mathematics

Sample solutions

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1. i. $P(A)=\{\varnothing,\{2\},\{\{2\}\},\{\{2,\{2\}\}\},\{2,\{2\}\},\{2,\{2,\{2\}\}\},\{\{2\},\{2,\{2\}\}\},\{2,\{2\},\{2,\{2\}\}\}\}$.

   ii. No. $1\in\mathbb{N}$. But $1\notin\{m+n:m,n\in\mathbb{N}\}$ because there does not exist $m,n\in\mathbb{N}$ such that $m+n=1$.

2. Let x be an arbitrary object.

   i. Then $x\in A\cup\varnothing$ iff $x\in A$ or $x\in\varnothing$
   iff $x\in A$.
   Thus both the sets $A\cup\varnothing, A$ have the same objects.
   Hence $A\cup\varnothing=A$.

   ii. $x\in A\cap\varnothing$ iff $x\in A$ and $x\in\varnothing$
   iff $x\in\varnothing$.
   Thus both the sets $A\cap\varnothing$, $\varnothing$ have the same objects.
   Hence $A\cap\varnothing=\varnothing$.

3. i. No. clearly $1,2\in\mathbb{N}$ and $0<1\le 100$, $0<2\le 100$. Thus $1,2\in\{x\in\mathbb{N}:0<x\le 100\}$.
   It is clear that $1<\frac{3}{2}<2$. Since $\frac{3}{2}\notin\mathbb{N}$, $\frac{3}{2}\notin\{x\in\mathbb{N}:0<x\le 100\}$.
   Thus $\{x\in\mathbb{N}:0<x\le 100\}$ is not an interval.

   ii. Suppose $(x,y)\in A\times(B\cup C)$.
   Then $x\in A$ and $y\in B\cup C$.
   Thus $x\in A$ and $(y\in B$ or $y\in C)$.
   Therefore $(x\in A$ and $y\in B)$ or $(x\in A$ and $y\in C)$.
   Hence $(x,y)\in A\times B$ or $(x,y)\in A\times C$.
   So, $(x,y)\in(A\times B)\cup(A\times C)$.
   Hence $A\times(B\cup C)\subseteq(A\times B)\cup(A\times C)$.

   Now suppose $(x,y)\in(A\times B)\cup(A\times C)$.
   Then $(x,y)\in A\times B$ or $(x,y)\in A\times C$.
   Thus, $(x\in A$ and $y\in B)$ or $(x\in A$ and $y\in C)$.
   Hence $x\in A$, and $(y\in B$ or $y\in C)$.
   Thus $x\in A$ and $y\in B\cup C$.
   Therefore $(x,y)\in A\times(B\cup C)$.
   Hence $(A\times B)\cup(A\times C)\subseteq A\times(B\cup C)$.
   Thus $A\times(B\cup C)=(A\times B)\cup(A\times C)$.

4. i. $R_1 = \{(1,1),(1,3),(3,1),(3,3),(3,5),(5,3)\}$ and $X = \{1,3,5\}$.
R_1 is not reflexive because, $5 \in X$ and $(5,5) \notin R_1$.
R_1 is symmetric because, $(1,3),(3,1) \in R_1, (3,5),(5,3) \in R_1, (1,1) \in R_1, (3,3) \in R_1$.
R_1 is not antisymmetric because, $(1,3),(3,1) \in R_1$ and $1 \neq 3$.
R_1 is not transitive because, $(5,3),(3,5) \in R_1$ and $(5,5) \notin R_1$.

ii. Let $y \in Y$. $\frac{y}{y} = 1 \in \mathbb{Q}$. Thus yR_2y. Hence R_2 is reflexive.

Now let $y_1, y_2 \in Y$. Suppose $y_1R_2y_2$. Then $\frac{y_1}{y_2} \in \mathbb{Q}$. Hence $\frac{1}{y_1/y_2} \in \mathbb{Q}$.

Therefore $\frac{y_2}{y_1} \in \mathbb{Q}$. Thus $y_2R_2y_1$. Hence R_2 is symmetric.

Now let $y_1, y_2, y_3 \in Y$ and suppose that $y_1R_2y_2$ and $y_2R_2y_3$. Thus $\frac{y_1}{y_2}, \frac{y_2}{y_3} \in \mathbb{Q}$.

Therefore $\frac{y_1}{y_2} \cdot \frac{y_2}{y_3} \in \mathbb{Q}$. That is $\frac{y_1}{y_3} \in \mathbb{Q}$. So, $y_1R_2y_3$. Hence R_2 transitive.
Thus R_2 is an equivalence relation.

5. (a) Let $A \in X$. Then $A \subseteq A$. Thus $A \leq A$. Thus $\leq$ is reflexive.
Now let $A, B \in X$ and suppose that $A \leq B$ and $B \leq A$.
Then $A \subseteq B$ and $B \subseteq A$. Hence $A = B$.
Thus $\leq$ is antisymmetric.
Now let $A, B, C \in X$ and suppose that $A \leq B$ and $B \leq C$,
Then $A \subseteq B$ and $B \subseteq C$.
Hence $A \subseteq C$.
Thus $A \leq C$.
Thus $\leq$ is transitive.
Therefore, $\leq$ is a partial order on X.

(b) It is clear that $\{3\} \not\leq \{4\}$ and $\{4\} \not\leq \{3\}$ because $\{3\} \not\subseteq \{4\}$ and $\{4\} \not\subseteq \{3\}$.
Thus neither $\{3\}$ nor $\{4\}$ is a least element of $\{\{3\},\{4\}\}$. Also neither $\{3\}$ nor $\{4\}$ is a greatest element of $\{\{3\},\{4\}\}$. This completes the proof.

6. Let $x \in \mathbb{R} \setminus \{2\}$. Then $f(x)$ is defined. Hence f is a function on $\mathbb{R} \setminus \{2\}$. Also $f(x) \neq -1$ because if $f(x) = -1$, then $\frac{2+x}{2-x} = -1$, so $2 + x = -2 + x$, and so $4 = 0$ is a contradiction.
Hence f is a function from $\mathbb{R} \setminus \{2\}$ into $\mathbb{R} \setminus \{-1\}$.

Let $x_1, x_2 \in \mathbb{R} \setminus \{2\}$ and suppose that $f(x_1) = f(x_2)$. Thus $\frac{2+x_1}{2-x_1} = \frac{2+x_2}{2-x_2}$.
So, $(2+x_1)(2-x_2) = (2+x_2)(2-x_1)$. Thus, $4 + 2x_1 - 2x_2 - x_1x_2 = 4 + 2x_2 - 2x_1 - x_1x_2$.
Hence $4x_1 = 4x_2$.

Thus $x_1 = x_2$.
Hence f is an injection.

Now let $y \in \mathbb{R} \setminus \{-1\}$.

Thus $y+1 \neq 0$. Hence $\frac{2(y-1)}{1+y} \in \mathbb{R}$. Also $\frac{2(y-1)}{y+1} \neq 2$, because if $\frac{2(y-1)}{y+1} = 2$ then $y-1 = y+1$ and hence $2 = 0$, which is contradiction. Hence $\frac{2(y-1)}{y+1} \in \mathbb{R} \setminus \{2\}$. Observe that

$$f\left(\frac{2(y-1)}{y+1}\right) = \frac{2+\frac{2(y-1)}{y+1}}{2-\frac{2(y-1)}{y+1}} = \frac{y+1+y-1}{y+1-(y-1)} = y.$$

Thus f is a surjection.
Hence f is a bijection.

7. (a) $f([0,\infty)) = \{f(x) : x \in [0,\infty)\}$
$= \{x^2 : x \in [0,\infty)\}$
$= [0,\infty)$.

(b) Let $A = \{-1\}$ and $B = \{1\}$.
Then $A \cap B = \{-1\} \cap \{1\} = \varnothing$.
Also $f(A) = \{f(-1)\} = \{(-1)^2\} = \{1\}$ and $f(B) = \{f(1)\} = \{1^2\} = \{1\}$.
Thus $f(A) = f(B)$.

8. i. There exists a bijection from X into $\{1,2,3,4,5\}$.
ii. Define $h : X \cup Y \to \{1,2,3,\ldots,m+n\}$ by

$$h(r) = \begin{cases} f(r) & \text{if } r \in X \\ g(r)+n & \text{if } r \in Y. \end{cases}$$

9. i. Define $f : \mathbb{N} \to \{\{1\},\{2\},\{3\}\}$ by $f(1) = \{1\}$, $f(2) = \{2\}$, $f(n) = \{3\}$ for each $n \geq 3$.
Thus $\{\{1\},\{2\},\{3\}\}$ is countable.
ii. Observe that the set of even positive integers is $\{2n : n \in \mathbb{N}\}$.
Define $f : \mathbb{N} \to \{2n : n \in \mathbb{N}\}$ by $f(n) = 2n$, $n \in \mathbb{N}$.
Clearly f is defined on $\mathbb{N}$. Observe that for each $n, m \in \mathbb{N}$, $f(n) = f(m)$ implies $2n = 2m$, so $n = m$. Thus f is one-to-one.
Also it is clear that f is onto $\{2n : n \in \mathbb{N}\}$.
Thus f is a bijection.
This completes the proof.

10. i. *F*.
ii. *F*.