The Open University of Sri Lanka
B.Sc/B.Ed Degree Programme
Final Examination 2008/2009
Level 05 - Applied Mathematics
AMU 3186/AME 5186 – Quantum Mechanics

Duration: - Two and Half Hours.

Date: - 02.07.2009. **Time: - 01.30 pm. – 04.00 pm.**

Answer FOUR questions only.

01. (i) Define the following:
Eigenvalues,
Eigenfunctions,
Hermitian Operator.

(ii) If A and B are two Hermitian operators, then show that
(a) A^2 is Hermitian.
(b) $(AB+BA)$ and $i(AB-BA)$ are also Hermitian.
(c) If $[A,B]=i\lambda$ then λ is real.

(iii) If $\hat{p}=-i\hbar\frac{d}{dx}$, show that $[\hat{p},\hat{x}]=-i\hbar$ and use the Mathematical Induction to show that $[\hat{p},\hat{x}^n]=-in\hbar x^{n-1}$, when n is a positive integer.

(iv) Show that $\psi(x)=\exp(ikx)$ is an eigenfunction of the momentum operator $\hat{p}=-i\hbar\frac{d}{dx}$.

02. The state function $\psi(x)$ of a particle, free to move on a straight line ox, is given by

$$\psi(x)=e^{\frac{-x^2}{\Delta^2}+\frac{ipx}{\hbar}},$$

where Δ, p and $\hbar$ are constants.

(i) Find the expectation values of the position and momentum of the particle.

(ii) State Heisenberg's Uncertainty principle.
Find the uncertainty of the momentum and of the position of the particle.

(iii) Verify the Heisenberg's uncertainty principle.

03. A particle of mass m moves in the x direction under a potential $V(x)=\frac{1}{2}m\omega^2x^2$, where ω is a positive constant.

(i) If $\hat{H}$ is the Hamiltonian operator, show that

$$\hat{H}=\hbar\omega\left(\zeta\zeta^{*}+\frac{1}{2}\right),$$

where $\zeta=(2m\hbar\omega)^{-1/2}(m\omega x-i\hat{p}_x)$.

(ii) Obtain the following commutation relations,

(a) $\left[\zeta,\zeta^{*}\right]=-1$

(b) $\left[\hat{H},\zeta\right]=\hbar\omega\zeta$

(c) $\left[\hat{H},\zeta^{*}\right]=-\hbar\omega\zeta^{*}$

(iii) Obtain the lowest eigenvalue of H.

04. (i) Define the expectation value of a dynamical variable represented by an operator $\hat{A}$.

(ii) A Hermitian operator $\hat{A}$ is defined as one for which, for all normalizable functions f and g,

$$\int f^{*}\hat{A}g\,d\tau=\int(\hat{A}f)^{*}g\,d\tau.$$

Show that its expectation value is real.

(iii) The time independent Schrödinger equation is given by

$$\hat{H}\psi=i\hbar\frac{\partial\psi}{\partial t},$$

where $\hat{H}$ is the Hamiltonian operator. Show that, if $\hat{A}$ is Hermitian, then

$$\frac{\partial}{\partial t}\langle\hat{A}\rangle=\frac{i}{\hbar}\left\langle\left[\hat{H},\hat{A}\right]\right\rangle+\left\langle\frac{\partial\hat{A}}{\partial t}\right\rangle.$$

(iv) Hence obtain Ehrenfests's Theorem for a particle moving in a potential V,

$$\frac{\partial}{\partial t}\langle\hat{p}\rangle=-\langle\nabla V\rangle,$$

where $\hat{p}$ is the momentum operator. Give the physical interpretation of this result..

05. A particle of mass m and energy E moves in the positive x direction and meets a barrier. The potential energy $V(x)$ of the particle is given by,

$$V(x)=\begin{cases} 0 & ;x\leq 0 \\ V_0 & ;0\leq x\leq a \\ 0 & ;x>a \end{cases}.$$

(i) Write down the equations satisfied by the wave function $\psi(x)$ in each of the three regions and state clearly the boundary conditions which need to be satisfied by $\psi(x)$.

(ii) Find the transmission coefficient T.

06. Angular momentum of a particle is defined as a vector $\underline{L}$, such that $\underline{L}=\underline{r}\times\underline{p}$ where $\underline{p}$ is the momentum and $\underline{r}$ is the position vector of the particle with respect to a fixed origin O.

(i) Write down the Cartesian components L_x, L_y, L_z of the angular momentum operator.
Hence obtain the angular momentum operator in spherical polar co–ordinates.

(ii) Prove the following relations for the angular momentum operator.

(a) $[L_x, L_y]=i\hbar L_z$,

(b) $[L^2, L_z]=0$.

(iii) If $L_+=L_x+iL_y$ and $L_-=L_x-iL_y$, Prove the following results,

(a) $[L_z, L_+]=\hbar L_+$,

(b) $[L_+, L_-]=2\hbar L_z$,

(c) $[L^2, L_+]=0$.