The Open University of Sri Lanka
B.Sc./B.Ed Degree Programme – Level 04
Final Examination 2010/2011
Applied Mathematics
APU 2143 – **Vector Calculus**
Duration :- Two Hours.

Date :-30.12.2010 **Time:- 1.00 p.m. - 3.00p.m.**

Answer Four Questions Only.

1. **(a)** Define the limit of a two variable function $f(x, y)$ as $(x, y) \to (a, b)$.

   (b) Define the continuity and differentiability of a two variable function at the point (a, b).

   (c) Discuss the continuity and differentiability of the following functions.

   (i) $f(x, y) = \begin{cases} \dfrac{xy^2}{x^2 + y^4} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

   (ii) $f(x, y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$

2. **(a)** If $\tan u = \dfrac{x^3 + y^3}{x - y}$, show that

   (i) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sin 2u$ and

   (ii) $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 2\cos 3u \sin u$.

   (b) Using linear approximations, obtain an approximate value for the real number $\sqrt{9(1.95)^2 + (8.1)^2}$.

   (c) Find the directional derivative of $f(x, y, z) = x^2yz + 4xz^2$ at $(1, -2, -1)$ in the direction $2\underline{i} - \underline{j} - 2\underline{k}$.

3. **(a) (i)** Define a stationary point of a single valued function $f(x, y)$ defined over the domain D. Explain briefly how you would determine its nature.

(ii) Find the maximum and minimum values of the function $f(x, y) = x^4 + y^4 - 4xy + 1$ and determine their nature.

(b) Use method of Lagrange multipliers to minimize the function $f(x, y) = x^2 - xy + 2y^2$ subject to $2x + y = 22$.

4. **(a)** Find the value of the surface integral of the function $f(x,y) = x^2 + y^2$ over the closed region bounded by $y = x$ and $y^2 = 4x$.

(b) Find the value of the surface integral of the function $f(x,y) = 4 - x^2 - y^2$ over the region in the first quadrant bounded by $x^2 + y^2 = 1$, $x^2 + y^2 = 9$, $y = 0$ and $x = 0$.

(c) Find the volume integral of the function $f(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$ over the hemispherical region bounded by the surfaces $x^2 + y^2 + z^2 = 3$ and $z = 0$.

5. **(a)** State Gauss' divergence theorem, giving the meanings of any symbols used.

(b) Verify Gauss' divergence theorem for the vector field $\underline{F} = x^3\underline{i} + y^3\underline{j} + z^3\underline{k}$, considering the region enclosed by the surface S of a sphere of radius R with centre at the origin.

(c) Find the divergences of the following vector fields.

(i) $\underline{r}$,

(ii) $r^n\underline{c}$ $(r \neq 0)$ where $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$, $r = |\underline{r}|$ and $\underline{c}$ is a constant vector.

6. **(a)** Define what is meant by a conservative vector field.

Show that the vector field $\underline{F} = 2xy^2ze^{-x^2z}\,\underline{i} - 2ye^{-x^2z}\,\underline{j} + x^2y^2e^{-x^2z}\,\underline{k}$ is irrotational, and determine the corresponding scalar potential ϕ such that $\underline{F} = \nabla\phi$.

(b) Given two functions $P(x, y)$ and $Q(x, y)$ defined in $\mathbb{R}^2$, prove Green's theorem

$$\oint_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy .$$

(i) Through an appropriate choice of P and Q, obtain an expression for the area of the region R.

(ii) Hence find the area of the region enclosed by $y = 3x^2$ and $y = 6x$.