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The Open University of Sri Lanka
B.Sc./B.Ed. Degree Programme-2010/2011
Pure Mathematics-Level 04
Final Examination
Sequences and Series-PUU2140

Duration: Two Hours

Date: 21.12.2010 Time: 9.30am-11.30am

Answer Four questions Only

1. (a) State the ε- N definition of limit of a sequence.

   Using the definition show that $\lim_{n\to\infty}\frac{n^3+2}{2n^3+7}=\frac{1}{2}$.

   (b) Show that the limit of a convergent sequence is unique.

   (c) Show that a convergent sequence is bounded.

   The Fibonacci numbers are recursively defined by $x_1=1, x_2=1,$ and $x_n=x_{n-1}+x_{n-2}$ for all $n>2$. Show that the sequence of Fibonacci numbers $\{1,2,3,5,\ldots\}$ does not converge.

2. (a) Let $\langle a_n\rangle$ be a monotone increasing sequence that is bounded above. Show that the sequence $\langle a_n\rangle$ converges.

   Let $a_n=\sum_{k=0}^{n}\frac{1}{k!}$ converges for $n\in\mathbb{N}$. Show that the sequence $\langle a_n\rangle$ is convergent.

   (b) Find $\lim_{n\to\infty}a_n$ where $\langle a_n\rangle$ is defined by

$$a_1=2$$
$$a_{n+1}=\frac{1}{2}(a_n+6)\quad \text{for all n>1.}$$

3. (a) State the definition of a Cauchy sequence.
Show that every convergent sequence is Cauchy.

Let $\langle x_n \rangle$ be a sequence defined by

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \text{ for all } n \in \mathbb{N}.$$

Show that $\langle x_n \rangle$ is not a Cauchy sequence.
Is it convergent?

(b) Let $\langle x_n \rangle$ be a bounded sequence in $\mathbb{R}$ and $l \in \mathbb{R}$. Prove that $\lim_{n\to\infty} x_n = l$ if and only if $\limsup x_n = \liminf x_n$.

4. (a) Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ convergent series. Show that $\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$.

(b) Let $\alpha \in \mathbb{R}$. Prove that $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$ converges if and only if $\alpha > 1$.

Show that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges where as $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ converges.

(c) State and prove the Comparison Test.

5. (a) Determine whether each of the following series converges. In each case, justify your answer.

(i) $\sum_{n=1}^{\infty} \frac{n}{n^3 + n + 1}$

(ii) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^3 + n + 1}$

(iii) $\sum_{n=1}^{\infty} \frac{n!(2n)!}{(3n)!}$

(iv) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n} + \log\left(\sqrt{n} + 1\right)}$

(b) What does it means to say that a series is conditionally convergent?

Suppose that the series $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, and that sequences $\langle b_n \rangle$ and $\langle c_n \rangle$ are defined as follows:

$$b_n = \max(a_n, 0) = \frac{1}{2}(a_n + |a_n|)$$

$$c_n = \min(a_n, 0) = \frac{1}{2}(a_n - |a_n|).$$

Prove that $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$ both diverge.

06. (a) Show that the following series converges for any $a > 1$:

$$\sum_{n=1}^{\infty} \frac{n}{(n^2+1)\left(\log(n^2+1)\right)^a}.$$

(b) Determine the values of the real number x for which the following series converges:

$$\sum_{n=1}^{\infty} \frac{nx^n}{n^2+2}.$$

(c) Suppose that $a_n \geq 0$ and that the series $\sum_{n=1}^{\infty} a_n$ diverges. Prove that the series

$$\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$$

also diverges.