00107

The Open University of Sri Lanka.
B.Sc./B.Ed. Degree Programme.
Final Examination – 2009/2010.
Level 05-Applied Mathematics.
AMU 3186/AME 5186 –Quantum Mechanics

Duration :- Two and a Half Hours

Date :- 23-06-2010 **Time :- 1 .00 p.m. – 3.30 p.m.**

Answer Four Questions Only

(1) The state function $\varphi(x)$ of a Gaussian wave function is given by

$\varphi(x)=\left(\frac{a}{\pi}\right)^{\frac{1}{4}}\exp\left(-\frac{a}{2}x^2\right)$, where a and π are constants.

(a) Find the expectation values of

(i) the position x

(ii) square of the position

(iii) the momentum p

(iv) square of the momentum.

(b) Show that $\Delta\varphi\, x.\Delta\varphi\, p=\frac{\hbar}{2}$.

(2) The conduction electrons in metals are held inside the metal by an average potential v called the inner potential of the metal. For the one dimensional model given by

$$v(x)=\begin{cases} v=-v_0, & x<0 \\ v=0, & x>0 \end{cases}$$

Calculate the probability of reflection and transmission of a conduction electron approaching the surface of the metal with total energy E for the case

(a) $E>0$

(b) $-v_0<E<0$

(3) Consider the Compton scattering of a photon of wave length λ_0 by a free electron moving with a momentum of magnitude P in the same direction as that of the incident photon.

(a) Show that in this case the Compton equation becomes

$$\Delta\lambda = \frac{2\lambda_0(p_0 + p)c}{E - pc}\sin^2\left(\frac{\theta}{2}\right).$$

where $p_o = \frac{h}{\lambda_o}$ is the magnitude of the incident photon moment, θ the scattering angle, h the plank constant, c the velocity of light and $E = (m^2c^4 + p^2c^2)^{\frac{1}{2}}$ is the initial electron energy.

(b) What is the maximum value of the electron momentum after the collision? Compare with the case $P=0$.

(c) If initially the free electron move with a momentum of magnitude P in a direction opposite to that of the incident photon, show that the Compton shift becomes

$$\Delta\lambda = \frac{2\lambda_0(p_0 - p)c}{E + pc}\sin^2\left(\frac{\theta}{2}\right).$$

(4) (i) Define the commutators of two operators A and B.

(ii) A, B, C and D are given operators, show the following.

(a) $\left[\hat{A}, +\hat{B}, \hat{C}\right] = \left[\hat{A}, \hat{C}\right] + \left[\hat{B}, \hat{C}\right]$

(b) $\left[\hat{A}, \hat{B}+\hat{C}+\hat{D}\right] = \left[\hat{A}, \hat{B}\right] + \left[\hat{A}, \hat{C}\right] + \left[\hat{A}, \hat{D}\right]$

(c) $\left[\hat{A}, \left[\hat{B}+\hat{C}\right]\right] = \hat{C}\left[\hat{A}, \hat{B}\right] + \hat{B}\left[\hat{C}, \hat{A}\right] = 0$

(d) $\left[\hat{A}\hat{B}, \hat{C}\right] = \hat{A}\left[\hat{B}, \hat{C}\right] + \left[\hat{A}, \hat{C}\right]\hat{B}$

(e) $\left[\hat{A}, \hat{B}\right] = -\left[\hat{B}, \hat{A}\right]$

(iii) If $p = i\hbar\frac{d}{dx}$ and H is the Hamiltonian operator show that $\left[\hat{p}, \hat{H}\right] = -i\hbar(\Delta v)$ and use the principle of mathematical induction to show that

$$\left[\hat{p}^n, \hat{H}\right] = -in\hbar x^{n-1}.$$

(5) If $\hat{A}$ is an operator corresponding to a quantum observable and $\langle\hat{A}\rangle$ is the corresponding expectation value, show that $\frac{d\langle\hat{A}\rangle}{dt} = \frac{1}{i\hbar}\left\langle\left[\hat{A}, \hat{H}\right]\right\rangle + \left\langle\frac{\partial\hat{A}}{\partial t}\right\rangle$, where $\hat{H}$ is the time independent Hamiltonian for the quantum system. A Hamiltonian motion in one dimension

is $\hat{H}=\frac{\hat{p}_x^{\,2}}{2m}-q\hat{x}$, where q is a constant. For the operators $\hat{r}$ and $\hat{p}_x$ which denote position and momentum respectively, show that the commutators

$$\left[\hat{r},\ \hat{H}\right]=\frac{1}{m}i\hbar\hat{p}_x$$
$$\left[\hat{p}_x,\ \hat{H}\right]=iq\hbar$$

Hence, show that, if at time $t=0$, $\langle r\rangle=0$ and $\langle p_x\rangle=0$, then

$$\langle\hat{p}_x(t)\rangle=qt \text{ and } \langle\hat{r}(t)\rangle=\frac{qt^2}{2m}.$$

(6) (i) For any two operators $\hat{A}$ and $\hat{B}$, show that,

$$\left[\hat{A},\ \hat{B}\right]\hat{B}+\hat{B}\left[\hat{A},\ \hat{B}\right]=\hat{A}\hat{B}^2-\hat{B}^2\hat{A}=\left[\hat{A},\ \hat{B}^2\right]$$

(ii) Angular momentum of a particle is defined as a vector $\underline{L}$ given by $\underline{L}=\underline{r}\times\underline{p}$, where $\underline{p}$ is the momentum and $\underline{r}$ is the position vector of the particle with respect to a fixed origin O and $\underline{p}$ is its momentum

(a) Write down the Cartesian components L_X, L_Y, L_Z of the angular momentum operator. Hence obtain the angular momentum operator in spherical polar co-ordinates (r,θ,π) where the coordinates axes are chosen in a standard manner

(b) Prove the following.

(I) $\left[\hat{L}_x,\ \hat{L}_x^{\,2}\right]=0$

(II) $\left[\hat{L}_x,\ \hat{L}^2\right]=0$