MPZ 3230 – Assignment # 04

Academic Year - 2006

Answer all questions,

1) Tabulate to three decimal places, the values of $\log_e x$ at <u>unit</u> intervals from x=1 to x=9 inclusive.

Use these values to evaluate $\int_1^9 \log_e x dx$

a) By the Trapezoidal rule
b) By Simpson's rule
c) Compare the above results with the correct value of the integral.

2) i) Find the sixth Taylor Polynomial for Cos2x, about 0.

ii) By using the above result evaluate $\int_{0.5}^{1} \frac{\sin 2x}{x^2}$ dx

3) a) Given a table of square roots, calculate $\sqrt{151}$ and $\sqrt{155}$, by appropriate Newton's interpolation formulas.

x	150	152	154	156
$y=\sqrt{x}$	12.247	12.329	12.410	12.490

b) Using the values given in the above table, calculate $\sqrt{153}$, by Lagrange's interpolation formula.

4) a) Three boxes B_1,B_2 and B_3 contain light bulbs. B_1 contains 10 bulbs of which 4 are defective, B_2 contains 6 bulbs of which 1 is defective and B_3 contains 8 bulbs of which 3 are defective .A bulb is drawn at random from any box. What is the probability that the bulb is defective?
(Your answer should contain the tree diagram as well)

b) The following data is gives the distribution of monthly expenditures per head of the residents of a village.

Monthly Expenditure (Rs.)	No. of Persons
3101-3500	10
3501-4000	15
4001-4500	14
4501-5000	12
5001-5500	13
5501-6000	14
6001-6500	15
6501-7000	17
7001-7500	20

i) Calculate mean, median and mode for the data.

ii) Calculate the standard deviation and Variance for the data.

5) a) The marketing manager of a Toy-manufacturing firm is planning to introduce a new toy into the market.40% of the toys introduced by the company have been successful, 60% have not been successful. Before the toy is actually marketed, market research is conducted and a report, either favourable or unfavourable , is complied 80% the success of toys received favourable report and 30% of the unsuccessful toys also received favourable reports. The marketing manager would like to know the probability that the new toy will be successful if it receives a favourable report. Find the answer for the above problem?

b) If x has the probability density function

$$f(x)=\begin{cases} ke^{-3x} & ; x>0 \\ 0 & ; \text{otherwise} \end{cases}$$

Find $P(0.5 \leq x \leq 1)$?

c) i) Find the Cumulative distribution function for the above random variable x.

ii) Using the above Cumulative distribution function, again calculate the probability given in part (b).

Please send your assignment on or before **25.11.2006** to the following address.
Please send your answer with your address (write back of your answer sheet) and when you are doing assignment use both sides of paper.

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MPZ 3230 – Model Answer – 04
Academic year – 2006

(01) Take $f(x) = \log_e x$

Since we have to unit intervals, $h = 1$

n	x_n	f_n
0	1	0
1	2	0.6931
2	3	1.0986
3	4	1.3863
4	5	1.6094
5	6	1.7918
6	7	1.9459
7	8	2.0794
8	9	2.1972

(a) By using the Trapezoidal rule,

$$\int_a^b f(x)dx = \frac{h}{2}\{f_0 + f_n + 2(f_1 + f_2 + \ldots + f_{n-1})\}$$

$$\int_1^9 \log_e x dx = \frac{1}{2}\{f_0 + f_8 + 2(f_1 + f_2 + \ldots + f_6 + f_7)\}$$

$$= \frac{1}{2}\left\{0 + 2.1972 + 2\begin{pmatrix}0.6931 + 1.0986 + 1.3863 \\ +1.6094 + 1.7918 + 1.9459 + 2.0794\end{pmatrix}\right\}$$

$$= \frac{1}{2}\{2.1972 + 2(10.6045)\}$$

$$= 11.7031$$

$$\therefore \int_1^9 \log_e x dx = 11.703$$

(b) By Simpson's rule

$$\int_a^b f(x)dx = \frac{h}{3}\{f_0 + f_n + 4(f_1 + f_3 + \ldots + f_{n-1}) + 2(f_2 + f_4 + \ldots + f_{n-2})\}$$

$$\int_1^9 \log_e x dx = \frac{1}{3}\{f_0 + f_8 + 4(f_1 + f_3 + f_5 + f_7) + 2(f_2 + f_4 + f_6)\}$$

$$= \frac{1}{3}\begin{Bmatrix}0 + 2.1972 + 4(0.6931 + 1.3863 + 1.7918 + 2.0794) \\ +2(1.0886 + 1.6094 + 1.9459)\end{Bmatrix}$$

$$= \frac{1}{3}\{2.1972 + 4(5.9506) + 2(4.6539)\}$$

$$= 11.7691$$

$$\int \log_e x \, dx = 11.769$$

(c) According to the exact method,

$$\int \log_e x \, dx = 11.7750$$

Absolute error between exact value and Trapezoidal method value
$= |11.730 - 11.775| = 0.045$

Absolute error between exact value and Simpson's rule value
$= |11.769 - 11.775| = 0.006$

Therefore Simpson's approximation is more accurate than the Trapezoidal approximation.

(02) (i) Let the sixth Taylor polynomial for cos 2x, about 0 be,

$$f(x) = f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{3!}f'''(0)x^3 + \frac{1}{4!}f^{iv}(0)x^4 + \frac{1}{5!}f^{v}(0)x^5 + \frac{1}{6!}f^{vi}(0)x^6$$

Let

$$\begin{aligned}
f(x) &= \cos 2x & &\Rightarrow f(0) = 1 \\
f'(x) &= -2\sin 2x & &\Rightarrow f'(0) = 0 \\
f''(x) &= -4\cos 2x & &\Rightarrow f''(0) = -4 \\
f'''(x) &= 8\sin 2x & &\Rightarrow f'''(0) = 0 \\
f^{iv}(x) &= 16\cos 2x & &\Rightarrow f^{iv}(0) = 16 \\
f^{v}(x) &= -32\sin 2x & &\Rightarrow f^{v}(0) = 0 \\
f^{iv}(x) &= -64\cos 2x & &\Rightarrow f^{vi}(0) = -64
\end{aligned}$$

Therefore sixth Taylor polynomial for cos 2x,

$$\cos 2x = 1 - 4\frac{x^2}{2!} + 16\frac{x^4}{4!} - 64\frac{x^6}{6!}$$

(ii) Since $\frac{d}{dx}(\cos 2x) = -2\sin 2x$

$$\therefore \sin 2x = -\frac{1}{2}\left(\frac{d}{dx}(\cos 2x)\right)$$

By using above result we know that,

$$\cos 2x \simeq 1 - 4\frac{x^2}{2!} + 16\frac{x^4}{4!} - 64\frac{x^6}{6!}$$

$$\therefore \sin 2x = -\frac{1}{2}\frac{d}{dx}\left(1-\frac{4x^2}{2!}+16\frac{x^4}{4!}-64\frac{x^6}{6!}\right)$$

$$= -\frac{1}{2}\left(0-8\frac{x}{2!}+64\frac{x^3}{4!}-384\frac{x^5}{6!}\right)$$

$$\therefore \sin 2x = 4\frac{x}{2!}-32\frac{x^3}{4!}+192\frac{x^5}{6!}$$

$$\therefore \frac{\sin 2x}{x^2} = \left(4\frac{x}{2!}-32\frac{x^3}{4!}+192\frac{x^5}{6!}\right)\times\frac{1}{x^2}$$

$$= \frac{4}{x}.\frac{1}{2!}-32\frac{x}{4!}+192\frac{x^3}{6!}$$

$$\therefore \int_{0.5}^{1}\frac{\sin 2x}{x^2}dx = \int_{0.5}^{1}\left(\frac{4}{x}.\frac{1}{2!}-32\frac{x}{4!}+192\frac{x^3}{6!}\right)dx$$

$$= \left[\frac{4}{2!}\ln x-\frac{32}{4!}\frac{x^2}{2}+\frac{192}{6!}\frac{x^4}{4}\right]_{0.5}^{1}$$

$$= \frac{4}{2!}[\ln x]_{0.5}^{1}-\frac{32}{4!}\left[\frac{x^2}{2}\right]_{0.5}^{1}+\frac{192}{6!}\left[\frac{x^4}{4}\right]_{0.5}^{1}$$

$$= 1.3862-0.50+0.0625$$

$$= 0.9487$$

(3) (a)

x	150	152	154	156
$y=\sqrt{x}$	12.247	12.329	12.410	12.490

With the above data, following table is constructed,

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
150	12.247			
		0.082		
152	12.329		-0.001	0
		0.081		
154	12.410		-0.001	
		0.080		
	12.490			
156				

(i) Since 151 lies near the beginning of the table, Newton's Forward interpolation formula is appropriate.
Then we get,

$$y_s = y_0 + \Delta y_0 \binom{s}{1} + \Delta^2 y_0 \binom{s}{2} + \text{---} + \Delta^n \binom{s}{n} y_0$$

$x = 151, \quad x_0 = 150, \quad h = 152 - 150 = 2$

$$s = \frac{x - x_0}{h} = \frac{151-150}{2} = 0.5$$

$$y(151) = 12.247 + 0.082(0.5) + \frac{(-0.001)(-0.25)}{2!} + \frac{0 \times (0.5)(0.5-1)(0.5-2)}{3!}$$

$$y(151) = 12.247 + 0.041 + 0.000125$$
$$= 12.288125$$

Since the given data is atmost correct to three decimal places, we get the required result, $\sqrt{151} = 12.288$

(ii) Since 155 lies near the end of the table, Newton's backward interpolation formula is appropriate.

Then we get

$$y_s = y_n + \Delta y_{n-1} \binom{s}{1} + \Delta^2 y_{n-2} \binom{s}{2} + \text{---} + (-1)^n \Delta^n y_0 \binom{s}{n}$$

$x = 155, \quad x_n = 156, \quad h = 2, \quad s = \frac{x - x_n}{h} = \frac{155-156}{2} = -0.5$

$$y(155) = 12.490 + 0.080(-0.5) + (-0.001)\frac{(-0.5)(-0.5-1)}{2!} + 0$$

$$y(155) = 12.490 - 0.04 - 0.000375$$
$$= 12.4496$$

Therefore $\sqrt{155} = 12.450$

(b) Lagrange's interpolation formula,

$$L(x) = y_0 L_0^{(n)}(x) + y_1 L_1^{(n)}(x) + \ldots + y_n L_n^{(n)}(x)$$

Where

$$L_j^{(n)} x = \frac{(x - x_0)\ldots(x - x_{j-1})(x - x_{j+1})\ldots(x - x_n)}{(x_j - x_0)\ldots(x_j - x_{j-1})(x_j - x_{j+1})\ldots\ldots(x_j - x_n)}$$

Here $n = 3$

$$\text{Then } L(153) = \frac{(153-150)(153-154)(153-156)}{(150-152)(150-154)(150-156)} \times 12.247 +$$

$$\frac{(153-150)(153-154)(153-156)}{(152-150)(152-154)(152-156)}\times 12.329+$$

$$\frac{(153-150)(153-152)(153-156)}{(154-150)(154-152)(154-156)}\times 12.410+$$

$$\frac{(153-150)(153-152)(153-154)}{(156-150)(156-152)(156-154)}\times 12.490$$

$$L(153)=-0.765437+6.935062+6.980625-0.780625$$

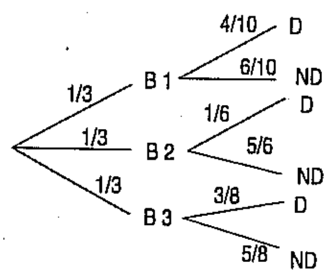
$$=\ 12.369625$$

Since the data are correct upto 3-decimal places, rounding the interpolated value correct upto 3 decimal places,

Therefore $\sqrt{153}\ \simeq\ 12.370$

(4) (a) In this problem there are 2 stages in the experiment. First, the selecting a box, then drawing a bulb which is either defective (D) or non defective (ND)

Tree diagram for the problem is,



Probability that a selected bulb being defective

$$P(D)=P(D/B_1).P(B_1)+P(D/B_2).P(B_2)+P(D/B_3).P(B_3)$$

$$=\ \frac{4}{10}\times\frac{1}{3}+\frac{1}{6}\times\frac{1}{3}+\frac{3}{8}\times\frac{1}{3}$$

$$=\ \frac{2}{15}+\frac{1}{18}+\frac{1}{8}$$

$$=\ 0.1333+0.0556+0.125$$

$$=\ 0.3139$$

(b)

Class	f_i	m_i	f_im_i	$d_i = \frac{m_i - 5750.50}{500}$	f_id_i	$f_id_i^2$
3101 – 3500	10	3300.5	33005.00	-4.9	-49	240.1
3501 – 4000	15	3750.5	56257.50	-4	-60	240
4001 – 4500	14	4250.5	59507.00	-3	-42	126
4501 – 5000	12	4750.5	57006.00	-2	-24	48
5001 – 5500	13	5250.5	68256.50	-1	-13	13
5501 – 6000	14	5750.5	80507.00	0	0	0
6001 – 6500	15	6250.5	93757.50	1	15	15
6501 – 7000	17	6750.5	114758.50	2	34	68
7001 - 7500	20	7250.5	145010.00	3	60	180
	130		708065.00		-79	930.1

(i) $\text{mean} = \frac{\sum_{i=1}^{n} f_i m_i}{n} = \frac{708065.00}{130} = 5446.65$

$$\text{median} = L + \left[\frac{\frac{n}{2} - F_1}{f}\right] \times w$$

f – frequency of median class = 14
w = true class width = 5500.5 – 5000.5 = 500

$$L = \frac{5501 + 5500}{2} = 5500.5$$

n = 130

F_1 = Cumulative Frequency of Class before median class = 64

$$\text{median} = 5500.5 + \frac{\left[\frac{130}{2} - 64\right]}{14} \times 500$$

$= 5500.5 + 35.7143$
$= 5536.21$

$$\text{Mode} = L + \left(\frac{d_1}{d_1 + d_2}\right) w$$

Model class = 7001 – 7500
Class width = w = 7500.5 – 7000.5 = 500
$d_1 = 20 - 17 = 3$
$d_2 = 20 - 0 = 20$

$$L = \frac{7001 + 7000}{2} = 7000.5$$

$$\text{mode} = 7000.5 + \left(\frac{3}{3+20}\right)500$$
$$= 7000.5 + 65.2174$$
$$= 7065.72$$

(ii) The standard deviation $= S = \sqrt{\frac{\sum_{i=1}^{n} f_i m_i^2}{n} - (\bar{x})^2}$

$$S = C\sqrt{\frac{\sum_{i=1}^{n} f_i d_i^2}{n} - (\bar{d})^2}$$

Where $d_i = \frac{m_i - A}{c}$ and $\bar{d} = \frac{\sum_{i=1}^{n} f_i d_i}{n}$

A – class midpoint somewhere in the middle of the table = 5750.5
C – most common class width = 500

$$S = 500\sqrt{\frac{930.1}{130} - \left(\frac{-79}{130}\right)^2} = 1302.43$$

$$\text{Variance} = S^2 = (1302.43)^2 = 1696323.90$$

(5) Let,

S – Toys introduced by the company been successful

S ' - Toys introduced by the company been not successful

F – Toys received a favourable report

Therefore according to the problem,

$P(S) = 0.4$, $P(S') = 0.6$, $P(F/S) = 0.8$, $P(F/S') = 0.3$

We want to know the probability, that the toy will be successful if it receives a favourable report.

$\therefore$ Required probability = P(S/F)

According to the Bayes' theorem, $P(S/F) = \frac{P(F/S).P(S)}{P(F/S).P(S) + P(F/S').P(S')}$

$$= \frac{0.8 \times 0.4}{(0.8 \times 0.4) + (0.3 \times 0.6)}$$

$$= \frac{0.32}{0.32+0.18} \qquad = 0.64$$

(b) $$f(x)=\begin{cases} ke^{-3x} & ;x>0 \\ 0 & ;\text{otherwise} \end{cases}$$

First we have to find the constant k,

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\therefore \underbrace{\int_{-\infty}^{0} f(x)dx}_{=0} + \int_{0}^{\infty} f(x)dx = 1$$

$$\therefore \int_{0}^{\infty} f(x)dx = 1$$

$$\therefore \int_{0}^{\infty} ke^{-3x}dx = 1$$

$$k\left[\frac{e^{-3x}}{-3}\right]_{0}^{\infty} = 1$$

$$\therefore k = 3$$

To find $P_r(0.5 \le x \le 1)$

$$P_r(0.5 \le x \le 1) = \int_{0.5}^{1} 3e^{-3x}dx = 3\left[\frac{e^{-3x}}{-3}\right]_{0.5}^{1} = 0.173$$

(c) (i) Let f(x) be the cumulative distribution function,

$$\therefore F(x) = \int_{-\infty}^{x} f(t)dt = \underbrace{\int_{-\infty}^{0} f(t)dt}_{0} + \int_{0}^{x} f(t)dt$$

$$= 3\left[\frac{e^{-3t}}{-3}\right]_{0}^{x} = -e^{-3x} + e^{0};\ x>0$$

$$= 1-e^{-3x};\ x>0$$

$$\therefore F(x) = \begin{cases} 1-e^{-3x} & ;x>0 \\ 0 & ;x\le 0 \end{cases}$$

(ii) $$P_r(0.5 \le x \le 1) = F(1) - F(0.5)$$

$$= \left(1-e^{-3}\right)-\left(1-e^{-1.5}\right)$$

$$= e^{-1.5} - e^{-3}$$

$$= 0.173$$