The Open University of Sri Lanka
B.Sc/B.Ed. Degree Programme
Final Examination- 2011/2012
Applied Mathematics-Level 05
APU3150/AMU3181/AME5181 – Fluid Mechanics

Duration:-Two hours

Date:-17.11.2012 **Time:-9:30a.m.-11:30a.m.**

Answer **FOUR** questions only.

1. Assuming the **general form of continuity equation** for a moving fluid, show that the velocity potential Φ, in irrotational motion of an incompressible fluid, satisfies Laplace's equation $\nabla^2\Phi = 0$.
In terms of **cylindrical polar coordinates** (r,θ,z), a velocity vector $\underline{\mathbf{q}}$ has components
$q_r = -U\left(1-\frac{a^2}{r^2}\right)\cos\theta,\ q_\theta = U\left(1+\frac{a^2}{r^2}\right)\sin\theta,\ q_z = 0,\ r \ge a,\ 0 \le \theta < 2\pi,$ where U and a are positive constants.
(i) Verify that $\underline{\mathbf{q}}$ represents the velocity vector in irrotational motion of an incompressible fluid, and find the velocity potential Φ.
(ii) Show that the equations of streamlines are given by $\left(r-\frac{a^2}{r}\right)\sin\theta = b,\ z = c,$ where b and c are constants, and locate the points of *stagnation.*
(iii) Identify the constant b and sketch the streamlines in the plane $z = 0$.

2. **Derive** *Euler's equation of motion* $\underline{\mathbf{F}} - \frac{1}{\rho}\,grad\,p = \frac{D\underline{\mathbf{q}}}{Dt}$, in the usual notation, for a perfect fluid, and deduce that the fluid is of constant density and the motion steady under *no external forces*, this equation can be re-written in the form $grad\left(\frac{p}{\rho}+\frac{1}{2}q^2\right) = \underline{\mathbf{q}} \times curl\underline{\mathbf{q}}$. Given further that the velocity vector $\underline{\mathbf{q}}$ is given as $\underline{\mathbf{q}} = \begin{cases} \omega r\,\underline{\mathbf{e}}_\theta, & 0 \le r < a \\ \frac{\omega a^2}{r}\underline{\mathbf{e}}_\theta, & r \ge a, \end{cases}$
referred to **cylindrical polar coordinates** (r,θ,z), where ω is a positive constant, find the **vorticity vector** and identify the type of motion, in each region.
If the pressure at infinity is p_∞, use the above equation of motion to find the pressure distribution in the region $r \ge a$.
Show that the pressure in the inner region is $p_\infty + \frac{\rho}{2}\omega^2\left(r^2 - 2a^2\right)$.

What is the minimum value of p_∞ which ensures positive pressure everywhere?

[Turn over

3. Infinite inviscid liquid of constant density is acted upon by a force $\underline{\mathbf{F}}=\left(\lambda r^{-k}\right)\left(-\underline{\mathbf{e}}_r\right)$, per unit mass, where λ and k are positive constants, $k>1$ and r is the distance of any point P from a fixed origin O and $\underline{\mathbf{e}}_r$ is the unit vector along $\overline{OP}$.

Find the scalar potential Ω such that $\underline{\mathbf{F}}=-grad\Omega$, assuming that $\Omega\to 0$ as $r\to\infty$.

Initially the liquid is at rest, and there is cavity whose boundary is the sphere $r=a$. In the subsequent motion of the liquid the pressure remains zero both at infinity and inside the cavity. If the radius of the cavity at time t is $R(t)$, show that the liquid velocity at point P at time t is $\underline{\mathbf{q}}=\left(\dfrac{R^2\dot{R}}{r^2}\right)\underline{\mathbf{e}}_r$, and find the velocity potential ϕ for the motion.

Using **Bernoulli's equation** show that the differential equation for $R(t)$ can be written as

$\dfrac{d}{dt}\left(R^{3/2}\dot{R}\right)=-2\lambda$, when $k=\dfrac{3}{2}$.

By direct integration of this equation, or otherwise, show that, with this value of k, the total time taken by the liquid to fill up the cavity will be $T=a^{5/4}\sqrt{\dfrac{2}{5\lambda}}$.

4. A uniform solid sphere of mass M and radius a moves in a straight line with velocity V through an infinite liquid which is at rest at infinity where the pressure is p_∞. Obtain the velocity potential $\phi=\dfrac{Va^3}{2r^2}\cos\theta$, suitably defined **spherical polar coordinates** (r,θ,ω), and establish the following results:

(i) The liquid in contact with points on the great circle of the surface whose plane is perpendicular to the direction of motion has a velocity $\dfrac{1}{2}V$, relative to the sphere.

(ii) The kinetic energy of the liquid is $\dfrac{1}{4}M'V^2$, where M' is the mass of the liquid displaced by the sphere.

(iii) The force acting on the sphere is $F=\left(M+\frac{1}{2}M'\right)\dfrac{dV}{dt}$.

5. **Derive** the velocity potential ϕ_0 at a point P whose **spherical polar coordinates** are (r,θ,ω), due to a uniform stream $-U\underline{\mathbf{i}}$, where $\underline{\mathbf{i}}$ is a unit vector along the axis $\theta = 0$.

Show that the velocity potential ϕ_1 at point P due to an isolated doublet of vector moment $\mu\underline{\mathbf{i}}$ placed at the origin O, is given by $\phi_1 = \mu\left(\frac{\cos\theta}{r^2}\right)$, fluid being at rest at infinity.

In the fluid motion represented by the velocity potential $\Phi = \phi_0 + \phi_1$, find the velocity components, and show that there is no flow of fluid across a certain spherical surface $r = a$, to be determined in terms of U and μ.

Taking this surface $(r = a)$ as a rigid boundary, re-write the expressions for the velocity components, involving the constants U and a (eliminating μ).

Locate the points on the sphere where the pressure takes *greatest* and *least* values, and show that these two values differ by an amount $\frac{9\rho U^2}{8}$.

6. Find the complex potential w_1, at a point $P(z = x + iy)$ due to a two-dimensional uniform stream $-U\underline{\mathbf{i}}$, where $\underline{\mathbf{i}}$ is the unit vector along the positive Ox-axis.

Write down the complex potential w_2, for a two-dimensional, isolated doublet of strength μ, whose axis makes an angle α with the positive Ox-axis, placed at point $P_0(z = z_0)$, when no boundary is present.

A two-dimensional doublet of strength $\mu\underline{\mathbf{i}}$ is at the point $z = ia$ in a stream of velocity $-U\underline{\mathbf{i}}$ in a semi-infinite liquid of constant density occupying the region $y \geq 0$ as a rigid boundary.
Show that

(i) the complex potential of the motion is $w(z) = Uz + \frac{2\mu z}{(z^2 + a^2)}$,

(ii) there are **no stagnation points** on the boundary, if $0 < \mu < 4a^2U$,

(iii) pressure on the boundary is least at the origin and greatest at the point $z = \pm a\sqrt{3}$.
