The Open University of Sri Lanka

B.Sc./B.Ed. Degree Programme

Final Examination-2011/2012

APU 2144/APE4144- Applied Linear Algebra and Differential Equations

APPLIED MATHEMATICS-LEVEL 04

Duration: Two Hours.

Date: 04.12.2012	**Time: 09.30 a.m. - 11.30 a.m.**

Answer FOUR questions only.

1. (i) Define the eigen values and eigen vectors of a given matrix.

   (ii) Let an eigen value of A be $\lambda(\neq 0)$. Then show that an eigen value of A^{-1} is $\frac{1}{\lambda}$.

   (iii) Find the eigen values of A where $A=\begin{pmatrix}2 & 1 & -1\\ 1 & 3 & 1\\ 1 & 2 & 2\end{pmatrix}$. Hence find the eigen values of A^{-1}.

   (iv) Determine the algebraic and geometric multiplicity of A where

$$A=\begin{pmatrix}1 & 2 & 2\\ 0 & 2 & 1\\ -1 & 2 & 2\end{pmatrix}.$$

2. (i) Determine the values of b for which the following system of equations has non-trivial solutions, and find them.

$$\begin{aligned}2x+3by+3b+4z&=0,\\ x+4y+5b+2z&=0,\\ x+y+5b+4z&=0.\end{aligned}$$

(ii) (a) State the Cayley-Hamilton theorem.

(b) Use the Cayley-Hamilton theorem to determine

$$A^{-1},\ A^{-2},\ A^{-3} \text{ if } A=\begin{pmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{pmatrix}.$$

(iii) Determine a, b and c so that A is orthogonal, where

$$A=\begin{pmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{pmatrix}.$$

(iv) Find a real symmetric matrix C such that $Q = X^T CX$ for each of the following quadratic forms.

(a) $Q = 6x_1^2 - 4x_1x_2 + 2x_2^2$,

(b) $Q = (x_1 + x_2 + x_3)^2$,

(c) $Q = 4x_1x_3 + 2x_2x_3 + x_3^2$.

3. (i) Find the general solution of each of the system of simultaneous differential equations, given below.

(a) $\dot{x}_1 = 7x_1 - x_2 + 6x_3$

$\dot{x}_2 = -10x_1 + 4x_2 - 12x_3$

$\dot{x}_3 = -2x_1 + x_2 - x_3$

(b) $\dot{x}_1 = 3x_1 + x_2 - x_3$

$\dot{x}_2 = x_1 + 3x_2 - x_3$

$\dot{x}_3 = 3x_1 + 3x_2 - x_3$

(ii) Solve the following systems of differential equations given below.

(a) $\dot{x}_1 = x_1 + 2x_2 + 6e^t$

$\dot{x}_2 = 3x_1 + 2x_2 - 6e^{2t}$

(b) $\dot{x}_1 = 3x_1 + x_2 - 2\sin t$

$\dot{x}_2 = 4x_1 + 3x_2 + 6\cos t$

4. (i) Find the general solution of each of the differential equations given below.

(a) $x^2\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+6y=4x-6$

(b) $x^2\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y=4\ln x$

(ii) Solve the boundary value problem,

$$u''(x)+\pi^2u(x)=0, \qquad u(0)=u(1), \quad u'(0)=u'(1).$$

(iii) Find all the eigen values and eigen functions for the boundary value problem $y''+\lambda y=0,\quad y'(0)=0,\ y'(2\pi)=0.$

5. (i) A change in coordinates from (x,y) to (ς,ϕ) is defined by $\varsigma=x^2+y$, $\phi=x^2-y$. Given that u is a function of two variables x and y, find $\frac{\partial^2 u}{\partial x^2}$ in terms of partial derivatives of u with respect to ς and ϕ.

(ii) If the density ρ is a function of x and y, show that under the transformation $\zeta=x^2-y^2,\ \phi=2xy$, the first order partial differential equation $x\frac{\partial\rho}{\partial x}-y\frac{\partial\rho}{\partial y}=0$ becomes $\frac{\partial\rho}{\partial\zeta}=0$, provided $x^2+y^2\neq 0$. Hence find ρ as a function of x and y.

(iii) Solve the partial differential equation $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}-3u=0$ with $u=x^2$ when $y=1$.

6. (i) Suppose that f and g are arbitrary single variable functions. Show that the function $u=f(x^2-y)+g(x^2+y)$ satisfies the equation

$$\frac{\partial^2 u}{\partial x^2}-\frac{1}{x}\frac{\partial u}{\partial x}-4x^2\frac{\partial^2 u}{\partial y^2}=0 \qquad (x\neq 0).$$

(ii) Use the change of variables $\varsigma=x^2+y^2$ and $\phi=y$, to find the general solution of the equation

$$y^2\frac{\partial^2 u}{\partial x^2}-2xy\frac{\partial^2 u}{\partial x\partial y}+x^2\frac{\partial^2 u}{\partial y^2}-\frac{y^2}{x}\frac{\partial u}{\partial x}-\frac{x^2}{y}\frac{\partial u}{\partial y}=0 \qquad (x\neq 0,\ y\neq 0).$$

(iii) Use the method of characteristics to solve the equation

$$y^2\frac{\partial^2 u}{\partial x^2}-x^2\frac{\partial^2 u}{\partial y^2}-y^2\left(4x+\frac{1}{x}\right)\frac{\partial u}{\partial x}+x^2\left(4y+\frac{1}{y}\right)\frac{\partial u}{\partial y}=0$$

in the region $x>0,\ y>0$.