00108

The Open University of Sri Lanka
B.Sc./B.Ed. Degree Programme-2011/2012
Final Examination
Pure Mathematics
PUU2144/PUE4144
Group Theory I

Duration; Two Hours

Date: 04.01.2012 Time: 9.30am-11.30am

Answer Four Questions Only.

1. (a) Let F be the set of all *fractional linear transformations* of the complex plane. That is, F is the set of all functions

$$f:\mathbb{C}\to\mathbb{C} \text{ with}$$
$$f(z)=\frac{az+b}{cz+d},$$

where the coefficients $a,\ b,\ c,\ d$ are integers with $ad\text{-}bc=1$. Show that F forms a group under composition of functions.

(b) Let G be a group. Let

$$C_G(x)=\{g\in G \mid gx=xg\}\subseteq G.$$

Prove that $C_G(x)\leq G$.

2. (a) Using regular polygon construct the Dihedral group D_8.

(b) Suppose that G is a group with the property that $x^2=1$, for $x\in G$. Show that G must be abelian.

(c) Does there exist a group G containing elements $a,\ b$ such that $a^2=b^2=(ab)^3=1$? Justify your answer.

3. (a) Let G be a finite group of even order. Show that G must contain an element of order 2.

(b) Suppose that G is a group and that $|G|=4$. By considering the orders of the elements of G (or otherwise), prove that G must be abelian.

(c) Find all subgroups of $\mathbb{Z}_{15}$ and draw the lattice diagram of subgroups.

4. (a) State and prove the Lagrange's theorem.

(b) Suppose that G is a group and that K, L are both normal subgroups with the property that $K \cap L = e$, where e is the identity element of K and L. Prove that every element of K commutes with every element of L.

(c) Let G be a group. Suppose that $H \leq G$ and that $|G:H| = 2$. Prove that $H \rhd G$.

5. (a) Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 3 & 1 & 4 & 6 & 5 & 7 & 8 & 10 & 9 \end{pmatrix}$ be a permutation.

(i) Write σ as a product of disjoint cycles.
(ii) Find the order of σ.
(iii) What is the inverse of σ.
(iv) If $\tau = (1 \ 2 \ 4 \ 8 \ 10 \ 5)(3 \ 7)$, find $\sigma \circ \tau$.

(b) State and prove the Class equation.

6. (a) Let G_1, G_2, H_1, H_2 be groups, and suppose that $\mu_1 : G_1 \to H_1$ and $\mu_2 : G_2 \to H_2$ are group isomorphisms. Define

$$\mu : G_1 \times G_2 \to H_1 \times H_2 \text{ by}$$
$$\mu(x_1, x_2) = (\mu_1(x_1), \mu_2(x_2))$$

for all $(x_1, x_2) \in G_1 \times G_2$. Prove that μ is a group isomorphism.

(b) State and prove the Second isomorphism theorem.

(c) State the Third isomorphism theorem.