The Open University of Sri Lanka
B.Sc./ B.Ed. Degree/ Continuing Education Programme
Level-04 Final Examination-2011/2012
PMU 2192/PME4192-Linear Algebra
Pure Mathematics

Duration: Two Hours.

Date: 11-01-2012.	**Time: 09.30 a.m. – 11.30 a.m.**

Answer FOUR questions only.

1. (i) Define the concept of a basis in a vector space.

(ii) Find a basis for the subspace W of $\Re^4$ spanned by

$$\left\{\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\2\end{pmatrix}, \begin{pmatrix}0\\2\\2\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\2\\2\\2\end{pmatrix}\right\}.$$

What is the dimension of W?

(iii) Determine whether each of the following are linear transformations:

(a) $L\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}=\begin{pmatrix}a_1\\a_2^2\end{pmatrix}$ (b) $L\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}=\begin{pmatrix}0\\3a_2\\a_3+2\end{pmatrix}$.

(iv) Let $L:\mathbb{R}^2\to\mathbb{R}^2$ be a linear transformation for which we know that

$$L\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}1\\-2\end{pmatrix},\ L\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}2\\3\end{pmatrix}.$$

(a) What is $L\begin{pmatrix}5\\18\end{pmatrix}$? (b) What is $L\begin{pmatrix}a_1\\a_2\end{pmatrix}$?

2. (i) Show that $\left(A^{-1}BA\right)^2 = A^{-1}B^2A$. Generalize to $\left(A^{-1}BA\right)^n$.

(ii) Let $A = \begin{pmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{pmatrix}$. Show that $|A| = (a+3b)(a-b)^3$.

(iii) Find the inverse of the matrix B where

$$B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(iv) Find non-singular matrices P and Q such that PAQ is of the normal form,

where $A = \begin{pmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{pmatrix}$.

Hence find the rank of A.

3. (i) The inner product space of continuous functions on [0, 1] is defined by

$$< f, g > = \int_0^1 f(t)g(t)dt.$$

Find $< f, g >$ in the following cases:

(a) $f(t) = t,\ g(t) = e^t$ (b) $f(t) = 2(t-1),\ g(t) = \dfrac{1}{t^2 - 2t + 3}$

(ii) Use Gram-Schmith orthogonalization process to construct an orthonormal set from

$$a_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},\quad a_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix},\quad a_3 = \begin{pmatrix} -1 \\ 0 \\ -2 \\ 1 \end{pmatrix}.$$

Justify your answer.

4. (i) Define the consistent and inconsistent systems.

(ii) If there are n homogeneous linear equations in n unknowns, then prove that the system has only the trivial solution if the co-efficient matrix is non-singular.

(iii) Find the rank of the matrix A where

$$A = \begin{pmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{pmatrix}.$$

(iv) Solve the following systems of simultaneous linear equations;

(a)
$$\begin{aligned} x+y+z &= 1 \\ x-y+z &= 3 \\ x \quad\quad +z &= k \end{aligned}$$

(b)
$$\begin{aligned} t+x+y+z &= 1 \\ t-x-y+z &= 0 \\ 2t+x+y-z &= 2. \end{aligned}$$

5. (i) State the Cramer's Rule.

(ii) Use the Cramer's Rule (if applicable) to find the solutions of the following system of linear equations.

$$\begin{aligned} x+2y-5z+2w &= -2 \\ 3x-y+2z+4w &= 19 \\ 2x+3y-4z+3w &= 6 \\ 5x+y+z-w &= 3 \end{aligned}$$

(iii) Use the Cayley Hamilton Theorem to compute A^{-1}, A^{-2}, A^{3} and A^{4} if

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{pmatrix}.$$

6. (i) Define the characteristic equation, the eigen values and the eigen vectors of a real square matrix A.

(ii) Let A be a matrix having eigen values λ_i where $i=1, 2, 3, \cdots$. Prove that the matrix kA has eigen values $k\lambda_i$ where k is a constant.

(iii) Let $A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$. Find an orthogonal matrix P such that

$P'AP$ is a diagonal matrix, where P' is the transpose of P.