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The Open University of Sri Lanka
B.Sc/ B.Ed Degree Programme – Level 05
Final Examination - 2011/2012
Pure Mathematics
PMU3294/ CSU3276/ PME5294–Discrete Mathematics

Duration: - Three Hours

Date: - 05-01-2012. **Time:- 9.30 a.m. – 12.30 p.m.**

Answer FIVE Questions only.

01. (a) Let p and q be two statements. Use the truth tables to determine whether each of the following statements is tautology, contradiction or contingency:

(i) $\left[\sim q\wedge(p\Rightarrow q)\right]\Rightarrow\sim p$,

(ii) $(p\wedge\sim q)\vee(q\wedge\sim p)$,

(iii) $p\wedge(p\Rightarrow q)\wedge\sim q$.

(b) Write the inverse and converse of each of the following statements:

(i) "If the density of a fluid is not $1000\,kg/m^3$ then the fluid cannot be water",

(ii) "If $\sqrt{2}$ is rational then either $\sqrt{2}$ is algebraic or $\sqrt{2}$ is transcendental".

(c) Let p be the statement "x is an even number" and let q be the statement "y is an odd number". Using logical connectives, write each of the following statements in symbolic form:

(i) "If x is an even number, then y is an odd number",

(ii) "x is an even number is a necessary condition to y is an odd number",

(iii) It is not the case that "x is an even number and y is not an odd number",

(iv) It is not the case that "x is an odd number only if y is an odd number",

(iv) "x is an even number if and only if y is an even number".

Hence write a logically equivalent statement to each of the above statements symbolically.

02. (a) Let a be any real number. Use the method of **proof by contradiction** to prove that "if $a^2 - 2a + 7$ is even, then a is odd".

Prove the same statement by the method of **proof by contraposition.**

(b) Prove or disprove each of the following statements:

(i) Let n be an integer. If $n > 4$ then $n^2 > 16$,

(ii) For each $x \in \mathbb{R}$, $(\log_e x)^2 = \log_e x^2$,

(iii) There exists $n \in \mathbb{N}$ such that $(n+2)^2 - (n+1)^2 = n^2$,

(iv) For each $n \in \mathbb{N}$, $5^{2n} + 12^{n-1}$ is divisible by 13.

(v) Let $a, b, c \in \mathbb{R}$. If $abc \neq 12$, then $a \neq 2$ or $b \neq 3$ or $c \neq 2$.

03. (a) Determine whether each of the following relations is an equivalence relation on $\mathbb{R}$.

(i) $R_1 = \{(x, y) \mid x, y \in \mathbb{R} \text{ and } |x - y| \leq 1\}$,

(ii) $R_2 = \{(A, B) \mid A, B \in P(\mathbb{R}) \text{ and } A \subset B\}$, where $A \subset B$ means that A is a proper subset of B and $P(\mathbb{R})$ is the set of all the subsets of $\mathbb{R}$.

(iii) $R_3 = \left\{(x, y) \mid x, y \in \mathbb{R} \text{ and } x - y = \frac{m}{2^n} \text{ for some } m \in \mathbb{Z} \text{ and } n \in \mathbb{Z}^+\right\}$.

(b) Show that if R and S are antisymmetric relations on a given set X, then so is $R \cap S$.

(c) List all the partial order relations on $X = \{a, b\}$.

Determine whether these are total orders.

04. (a) Let G be an abelian group with the binary operation $*$. Let $a, b \in G$ and n be a positive integer. Prove each of the following:

(i) $a * b^n = b^n * a$,

(ii) $(a * b)^n = a^n * b^n$,

(b) Let G be a set of all subsets of a universal set U and the binary operation $*$ for combining sets is defined as $A * B = (A \cup B) - (A \cap B)$, where $A, B \in G$. By assuming that the associative property is satisfied, show that G is a group under $*$.

If S is a subset of the set of natural numbers, solve the equation $\{1, 2, 4\} * S = \{3, 4\}$.

(c) Let G be the group of real numbers under addition, and let G' be the group of positive real numbers under multiplication. Show that the mapping $f: G \to G'$ defined by $f(a) = 2^a$ is a homomorphism.

Is it an isomorphism? **Justify your answer.**

05. (a) Let n be a positive integer greater than 2. Show that $n^2 + 1$ is a divisor of $(n+1)! - n! + (n-1)!$.

Deduce that $n^3 - n^2 + n - 1$ is also a divisor of $(n+1)! - n! + (n-1)!$.

(b) (i) Show that among any set of 51 positive integers less than 100, there is a pair whose sum is 100.

(ii) A teacher is making a multiple choice quiz. She wants to give each student the same questions, but have each student's questions appear in a different order. If there are twenty students in the class, what is the least number of questions the quiz must contain?

(iii) If two cards are chosen at random from a standard deck of playing cards, how many different ways are there to draw the two cards if at least one card is a jack, queen or a king?

(c) Find the number of arrangements of the letters of the word "ENGINEERING".

(i) In how many of them do the three E's come together at the beginning or the three N's come together at the end of the word.

(ii) How many different words with 3 letters can be made by selecting letters from the above word.

06. (a) Let A and B be any two events in a probability space S. Prove each of the following:

(i) $P(A\setminus B)=P(A)-P(A\cap B)$,

(ii) If $A\subseteq B$, then $P(A)\leq P(B)$,

(iii) If A and B are independent events, then A^C and B are also independent events, where A^c is the complement event of A.

(b) A fair coin is tossed thrice. Let A be the event of obtaining 2 consecutive heads in the first two tosses and let B be the event of obtaining a tail in the third toss.

(i) Find $P(A\cap B)$.

Are A and B independent events? **Justify your answer.**

(i) What is the probability of obtaining 2 consecutive heads in the first two tosses given that the third toss gave a tail?

(c) Let a die is loaded so that the probability of obtaining n points is proportional to n.
Find the probability of getting an even number when rolling this die.

07. (a) Let $G=G(V, E)$ be a graph of 5 vertices. By drawing the graph for each of the following sets of edges, determine whether the graph is connected:

(i) $E(G)=\{(x, y)|\ x+y \text{ is odd}\}$,

(ii) $E(G)=\{(x, y)|\ xy>5\}$.

(b) (i) Find the largest possible number of vertices in a graph with 30 edges if all the vertices have degree of at least 3.

(ii) Let the graph $G = G(V, E)$ has 5 vertices. Find the maximum number of edges in E.

(iii) Give an example of a graph such that every vertex is adjacent to 2 vertices and every edge is adjacent to 2 edges.

(c) Let G be a graph with set of four vertices $\{v_1, v_2, v_3, v_4\}$, whose adjacency matrix A is given by

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

(i) Show that G is connected,

(ii) Is G a forest? **Justify your answer,**

(iii) The subgraph H of G is defined by

$V(H) = \{v_1, v_3, v_4\}$ and $E(H) = \{\{v_1 v_3\}, \{v_1 v_4\}, \{v_3 v_4\}\}$.

Determine whether H is a component of G. **Justify your answer.**

08. (a) A person wants to climb up a staircase. At any given stage he can only take one or two steps up.

Let f_n be the number of ways he can climb the staircase if it has n steps.

(i) Find a difference equation satisfied by f_n and initial conditions, f_1 and f_2, that define the sequence $\{f_n\}$,

(ii) If the staircase has 10 steps, find the number of ways he can climb the staircase,

(iii) Use the method of induction to show that if the staircase has n steps, then he can climb the staircase in $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$ ways.

(b) Find the general solution of the difference equation $f(n+2) - 9f(n) = n(1+3^n)$.