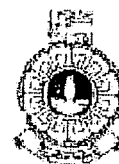


The Open University of Sri Lanka  
 B.Sc. /B.Ed. Degree Programme  
 Final Examination-2012/2013  
 Pure Mathematics - Level 05  
 PMU3295/PME5295 - Ring Theory



**Duration: Three Hours**

**Date: 25.11.2013**

**Time: 9.30 a.m. – 12.30 p.m.**

**Answer Five Questions Only.**

1.(a) Suppose  $F = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\} = \mathbb{Q}(\sqrt{2})$ .

Show that  $F$  is a commutative ring with unity under usual addition and multiplication.

(b) Prove each of the following properties for elements  $a$  and  $b$  of a ring  $R$ :

- (i)  $a0 = 0a = 0$ ,
- (ii)  $a(-b) = (-a)b = -(ab)$ ,
- (iii)  $a(-b) = (-a)(-b) = ab$ .

2. (a) (i) Prove that the set  $U = \{a \in R \mid ab = ba \text{ for each } b \in R\}$  in a given ring  $R$  is a commutative subring of  $R$ .

(ii) Is the subset  $S = \{n \mid n = 0 \text{ or } n \text{ is odd}\}$  of the ring  $(\mathbb{Z}; +, \cdot)$  a subring?

Justify your answer.

(b) For given subrings  $U_1$  and  $U_2$  of a ring  $R$ , show that their intersection  $U_1 \cap U_2$  is also a subring of  $R$ .

3. (a) For a given element  $a$  of a ring  $R$ , the set  $A$  is defined as follows:

$$A = \{ba + na \mid b \in R, n \in \mathbb{Z}\}$$

Prove each of the following:

- (i)  $A$  is a left ideal of  $R$ .
- (ii) If  $I$  is any left ideal of  $R$  with  $a \in I$ , then  $A \subset I$ .

(b) Let the ideal  $I = 7\mathbb{Z}$  in the ring  $(\mathbb{Z}; +, \cdot)$ . Show that either  $A = I$  or  $A \subset I$  if  $A$  is any ideal of  $\mathbb{Z}$  with  $I \subset A$ .

4. (a) Prove that a homomorphism  $\phi: R \rightarrow R'$  is one-one if and only if  $\text{Ker}\phi = \{0\}$ .

(b) Let  $R = (\mathbb{C}, +, \cdot)$  and  $R' = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a, b \in R \right\}$ .

The mapping  $\psi: R \rightarrow R'$  is defined for  $z = a + ib \in \mathbb{C}$  by  $\psi(z) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ .

Show that  $\psi$  is an isomorphism.

5. (a) State the First Isomorphism Theorem.

The mapping  $\varphi: (\mathbb{Z}; +, \cdot) \rightarrow (\mathbb{Z}_n; +_n, \cdot_n)$ , is defined by  $\varphi(m) = [m]$ ,  $m \in \mathbb{Z}$ .

where the binary operations  $+_n$  and  $\cdot_n$  on  $\mathbb{Z}_n$  are defined as addition modulo  $n$ , multiplication modulo  $n$  respectively.

(i) Show that  $\varphi$  is a homomorphism,

(ii) Find the Kernel of  $\varphi$ ,

(iii) Deduce that  $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$ .

(b) Show that an isomorphism between two rings is an equivalence relation.

6. Let  $R = R_1 \times R_2$  be the direct product of rings.

(a) (i) Show that  $R$  has unit elements if  $R_1$  and  $R_2$  have unit elements.

(ii) Show that  $R$  is commutative if  $R_1$  and  $R_2$  are commutative.

(b) Is it true that  $R$  is an integral domain if  $R_1$  and  $R_2$  are integral domains? Justify your answer.

7. Define the terms *prime ideal* and *maximal ideal* in a commutative ring  $R$  with identity.

(a) Let  $I$  be a proper ideal of the ring  $R$ .

Prove that,  $I$  is a prime ideal if and only if the quotient ring  $R/I$  is an integral domain.

(b) Prove or disprove the following statement:

In a commutative ring without the identity, every maximal ideal is a prime ideal.

8. (a) The binary operations  $\oplus$  and  $\otimes$  are defined on a set  $R$  by

$$a \oplus b = a + b - 1$$

$$a \otimes b = a + b - ab.$$

Show that  $(R, \oplus, \otimes)$  is a field.

(b) If the characteristic of a field  $F$  is prime  $p$ , then show that  $F$  contains a subfield isomorphic to  $(\mathbb{Z}_p; +_p, \cdot_p)$ .

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