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The Open University of Sri Lanka
B.Sc. /B.Ed. Degree Programme
Final Examination-2012/2013
Pure Mathematics - Level 05
PMU3295/PME5295 - Ring Theory

Duration: Three Hours

Date: 25.11.2013 **Time: 9.30 a.m. – 12.30 p.m.**

Answer Five Questions Only.

1.(a) Suppose $F=\{a+b\sqrt{2} \mid a,b\in\mathbb{Q}\}=\mathbb{Q}(\sqrt{2})$.
Show that F is a commutative ring with unity under usual addition and multiplication.

(b) Prove each of the following properties for elements a and b of a ring R:

(i) $a0=0a=0$,
(ii) $a(-b)=(-a)b=-(ab)$,
(iii) $a(-b)=(-a)(-b)=ab$.

2. (a) (i) Prove that the set $U=\{a\in R \mid ab=ba \text{ for each } b\in R\}$ in a given ring R is a commutative subring of R.

(ii) Is the subset $S=\{n \mid n=0 \text{ or } n \text{ is odd}\}$ of the ring $(R;+,\cdot)$ a subring?
Justify your answer.

(b) For given subrings U_1 and U_2 of a ring R, show that their intersection $U_1\cap U_2$ is also a subring of R.

3. (a) For a given element a of a ring R, the set A is defined as follows:
$$A=\{ba+na \mid b\in R,\ n\in\mathbb{Z}\}$$
Prove each of the following:
(i) A is a left ideal of R.
(ii) If I is any left ideal of R with $a\in I$, then $A\subset I$.

(b) Let the ideal $I=7\mathbb{Z}$ in the ring $(\mathbb{Z};+,\cdot)$. Show that either $A=I$ or $A\subset I$ if A is any ideal of $\mathbb{Z}$ with $I\subset A$.

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4. (a) Prove that a homomorphism $\phi : R \to R'$ is one–one if and only if $Ker\phi = \{0\}$.

   (b) Let $R = (\mathbb{C}, +, \cdot)$ and $R' = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a, b \in R \right\}$.

   The mapping $\psi : R \to R'$ is defined for $z = a + ib \in \mathbb{C}$ by $\psi(z) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.

   Show that ψ is an isomorphism.

5. (a) State the First Isomorphism Theorem.

   The mapping $\varphi : (\mathbb{Z}; +, \cdot) \to (\mathbb{Z}_n; +_n, \cdot_n)$, is defined by $\varphi(m) = [m]$, $m \in \mathbb{Z}$.

   where the binary operations $+_n$ and $\cdot_n$ on $\mathbb{Z}_n$ are defined as addition modulo n, multiplication modulo n respectively.

   (i) Show that φ is a homomorphism,

   (ii) Find the Kernal of φ,

   (iii) Deduce that $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$.

   (b) Show that an isomorphism between two rings is an equivalence relation.

6. Let $R = R_1 \times R_2$ be the direct product of rings.

   (a) (i) Show that R has unit elements if R_1 and R_2 have unit elements.

   (ii) Show that R is commutative if R_1 and R_2 are commutative.

   (b) Is it true that R is an integral domain if R_1 and R_2 are integral domains? Justify your answer.

7. Define the terms *prime ideal* and *maximal ideal* in a commutative ring R with identity.

   (a) Let I be a proper ideal of the ring R.

   Prove that, I is a prime ideal if and only if the quotient ring R/I is an integral domain.

   (b) Prove or disprove the following statement:

   In a commutative ring without the identity, every maximal ideal is a prime ideal.

8. (a) The binary operations $\oplus$ and Θ are defined on a set R by

   $a \oplus b = a + b - 1$

   $a \Theta b = a + b - ab$.

   Show that $(R, \oplus, \Theta)$ is a field.

   (b) If the characteristic of a field F is prime p, then show that F contains a subfield isomorphic to $(\mathbb{Z}_p; +_p, \cdot_p)$.

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