The Open University of Sri Lanka
B.Sc. /B.Ed. Degree Programme
Final Examination - 2013/2014
Pure Mathematics - Level 04
PUU2144/PUE4144 – Group Theory I

Duration: Two Hours

Date: 22.11.2014 **Time: 9.30 a.m. – 11.30 a.m.**

Answer Four Questions Only.

1. (a) Let G be a set of mappings where
$G=\{f_{ab} \mid \text{for given } a,\ b\in\mathbb{R},\ a\neq 0,\ f_{ab}:\mathbb{R}\to\mathbb{R} \text{ with } f_{ab}(x)=ax+b,\ x\in\mathbb{R}\ \}$.
Let a binary operation $*$ on G be defined by
$f_{ab}*f_{cd}=(ax+b)*(cx+d)=a(cx+d)+b$ for all $f_{ab}, f_{cd}\in G$.
Show that $(G,*)$ is a group.

   (b) If $a^2=e$ for each element a of a group G, where e is the identity element of G, show that G is commutative.

2. (a) Let G be a group. Let H and K be subgroups of G.
   (i) Prove that $H\cap K$ is a subgroup of G.
   (ii) If $|H|$ and $|K|$ are relatively prime integers, show that $H\cap K=\{e\}$.

   (b) Let the set $\xi(G)=\{a\in G \mid ga=ag\ \forall g\in G\}$ be a subset of any given group G.
   Show that $\xi(G)$ is a subgroup of G.

3. (a) Define a normal subgroup of a group G.
   Establish the equivalence of the following statements for a subgroup H of a given group G.
   (i) H is a normal subgroup of G,
   (ii) $gHg^{-1}=H$ for each $g\in G$,
   (iii) $gHg^{-1}\subseteq H$ for each $g\in G$,
   (iv) $ghg^{-1}\in H$ for each $g\in G$ and $h\in H$.

   (b) Show that if A is a subgroup of G and B is a normal subgroup of G then AB is a normal subgroup of G, where $AB=\{x \mid x=ab,\ a\in A,\ b\in B\}$

4. (a) (i) Prove that any cyclic group of order n has a unique subgroup of order m for each m that divides n.

   (ii) Find all subgroups of $\mathbb{Z}_{18}$.

   (iii) Determine the subgroup lattice for $\mathbb{Z}_{18}$.

   (b) Let G be the cyclic group of order 4 generated by a. Let $H = \langle a^2 \rangle$.

   (i) Find all right cosets of H in G.

   (ii) Prove that the union of these cosets is G.

5. (a) Let $(G,*)$ and $(G',*')$ be two groups.

   If $\phi : G \to G'$ is a one to one and onto homomorphism, then prove each of the following:

   (i) G is commutative if and only if G' is commutative.

   (ii) An element e of G is an identity element if and only if $\phi(e)$ is an identity element of G'.

   (iii) Let $x \in G$. An element y of G is an inverse of x in G if and only if $\phi(y)$ is an inverse of $\phi(x)$ in G'.

   (b) Let $G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc \neq 0,\ a,b,c,d \in \mathbb{R} \right\}$ with operation matrix multiplication.

   Let $\varphi : G \to \mathbb{R}^*$, the non zero real numbers, be defined by $\varphi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = ad - bc$.

   Prove that φ is a homomorphism from G onto the multiplicative group of nonzero real numbers, $\mathbb{R}^*$.

6. (a) State Fundamental Homomorphism Theorem.

   Let N be a normal subgroup of G and let H be a subgroup of G. Suppose that HN is a subgroup of G. Define the map ϕ by $\phi : H \to HN/N$.

   (i) Prove that N is a normal subgroup of HN.

   (ii) Show that ϕ is well defined.

   (iii) Prove that ϕ is an onto homomorphism.

   (iv) Deduce that $H/H \cap N \cong HN/N$.