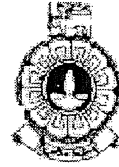


The Open University of Sri Lanka
 B.Sc. /B.Ed. Degree Programme
 Final Examination - 2013/2014
 Pure Mathematics - Level 04
 PMU2195/PME4195 – Theory of Integration



Duration: Two Hours

Date: 25.11.2014

Time: 1.30 p.m. – 3.30 p.m.

Answer Four Questions Only.

1. (a) Let f be a bounded function on $[a, b]$ and let P be a partition of $[a, b]$.
 Prove that $L(P, f) \leq U(P, f)$.
- (b) Let f be a function bounded on $[a, b]$. Suppose that P_1 and P_2 are partitions of $[a, b]$ such that P_2 is a refinement of P_1 and P_2 has only one extra point than P_1 .
 Prove that $U(P_1, f) \geq U(P_2, f)$.
- (c) Let f and g be two bounded functions defined on $[0, 2]$ and $P \in P[0, 2]$. Determine whether $L(P, fg) = L(P, f)L(P, g)$.

2. (a) Let $f(x) = k$ for $x \in [a, b]$, where k is a constant. Prove that $\int_a^b f(x) = k(b - a)$.

(b) Let $f(x) = \begin{cases} 2, & x \in [2, 3], \\ 3, & x \in (3, 4]. \end{cases}$ Show that $\int_{-2}^4 f(x) dx = 5$.

- (c) Let two functions f and g are defined on $[0, 1]$. Determine whether,

$$\int_0^1 (f + g)(x) dx = \int_0^1 f(x) dx + \int_0^1 g(x) dx.$$

3. (a) State Riemann's criterion.

(b) Let $f(x) = \begin{cases} 1, & 0 \leq x < \frac{1}{3}, \\ 2, & x = \frac{1}{3}, \\ 3, & \frac{1}{3} \leq x < 1. \end{cases}$

Use Riemann's criterion to show that f is Riemann integrable on $[0, 1]$.

(c) Let $f(x) = \begin{cases} 1, & x \in [0, 1] \cap \mathbb{Q}, \\ 0, & x \in [0, 1] \cap \mathbb{Q}^c. \end{cases}$

Use the Riemann's criterion to show that f is not Riemann integrable on $[0, 1]$.

4. (a) Let f be a bounded and monotonically increasing function on $[a, b]$. Prove that f is Riemann integrable on $[a, b]$.

(b) Let $f(x) = \sin x$ for $x \in [0, \pi/2]$. Prove that f is Riemann integrable on $[0, \pi/2]$.

(c) Is the following statement true? Justify your answer.

If f is a bounded function, which is not monotone on $[0, 1]$ (neither monotonically increasing nor monotonically decreasing on $[0, 1]$). Then f is not Riemann integrable.

5. (a) Let f be a function, continuous on a closed and bounded interval $[a, b]$. Prove that f is Riemann integrable on $[a, b]$.

(b) Suppose f is Riemann integrable on $[a, b]$ and $a < c < b$. Prove that

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

(c) Let $f(x) = \begin{cases} x, & x \in [0, 1], \\ x^2, & x \in (1, 2]. \end{cases}$ Evaluate $\int_1^3 f(x) dx$.

6. (a) State and prove the Mean-Value Theorem for integrals.

(b) State the Generalized Mean-Value Theorem and prove that the Generalized Mean-Value theorem implies the Mean-Value theorem.

(c) Determine the convergence of $\int_2^3 \frac{1}{3-x} dx$.