The Open University of Sri Lanka
B.Sc. /B.Ed. Degree Programme
Final Examination - 2013/2014
Pure Mathematics - Level 04
PMU2195/PME4195 – Theory of Integration

Duration: Two Hours

Date: 25.11.2014 **Time: 1.30 p.m. – 3.30 p.m.**

Answer Four Questions Only.

1. (a) Let f be a bounded function on $[a,b]$ and let P be a partition of $[a,b]$. Prove that $L(P,f) \le U(P,f)$.

   (b) Let f be a function bounded on $[a,b]$. Suppose that P_1 and P_2 are partitions of $[a,b]$ such that P_2 is a refinement of P_1 and P_2 has only one extra point than P_1. Prove that $U(P_1,f) \ge U(P_2,f)$.

   (c) Let f and g be two bounded functions defined on $[0,2]$ and $P \in \mathcal{P}[0,2]$. Determine whether $L(P,fg) = L(P,f)L(P,g)$.

2. (a) Let $f(x) = k$ for $x \in [a,b]$, where k is a constant. Prove that $\int_a^b f(x) = k(b-a)$.

   (b) Let $f(x) = \begin{cases} 2, & x \in [2,3], \\ 3, & x \in (3,4]. \end{cases}$ Show that $\int_2^4 f(x)dx = 5$.

   (c) Let two functions f and g are defined on $[0,1]$. Determine whether,

$$\int_0^1 (f+g)(x)dx = \int_0^1 f(x)dx + \int_0^1 g(x)dx.$$

3. (a) State Riemann's criterion.

   (b) Let $f(x) = \begin{cases} 1, & 0 \le x < \frac{1}{3}, \\ 2, & x = \frac{1}{3}, \\ 3, & \frac{1}{3} \le x < 1. \end{cases}$

   Use Riemann's criterion to show that f is Riemann integrable on $[0,1]$.

   (c) Let $f(x) = \begin{cases} 1, & x \in [0,1] \cap \mathbb{Q}, \\ 0, & x \in [0,1] \cap \mathbb{Q}^c. \end{cases}$

   Use the Riemann's criterion to show that f is not Riemann integrable on $[0,1]$.

4. (a) Let f be a bounded and monotonically increasing function on $[a,b]$. Prove that f is Riemann integrable on $[a,b]$.

   (b) Let $f(x) = \sin x$ for $x \in \left[0, \pi/2\right]$. Prove that f is Riemann integrable on $\left[0, \pi/2\right]$.

   (c) Is the following statement true? Justify your answer.

   If f is a bounded function, which is not monotone on $[0,1]$ (neither monotonically increasing nor monotonically decreasing on $[0,1]$). Then f is not Riemann integrable.

5. (a) Let f be a function, continuous on a closed and bounded interval $[a,b]$. Prove that f is Riemann integrable on $[a,b]$.

   (b) Suppose f is Riemann integrable on $[a,b]$ and $a < c < b$. Prove that $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

   (c) Let $f(x) = \begin{cases} x, & x \in [0,1], \\ x^2, & x \in (1,2]. \end{cases}$ Evaluate $\int_1^3 f(x)dx.$

6. (a) State and prove the Mean-Value Theorem for integrals.

   (b) State the Generalized Mean-Value Theorem and prove that the Generalized Mean-Value theorem implies the Mean-Value theorem.

   (c) Determine the convergence of $\int_2^3 \frac{1}{3-x} dx.$