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The Open University of Sri Lanka
B.Sc/B.Ed. Degree Programme
Final Examination - 2013/2014
Pure Mathematics - Level 04
PUU2143/PUE4143 – Differentiable Functions
Duration:-Two hours

Date:-03-12-2014 **Time:- 01:30 p.m.-03:30 p.m.**

Answer **FOUR** questions only.

1. (a) State the $\varepsilon-\delta$ definition for differentiability of a real valued function at a given point.

Let $f:\mathbb{R}\to\mathbb{R}$ be given by $f(x)=\begin{cases} x^2\sin\left(\dfrac{1}{x}\right), & x\neq 0 \\ 0, & x=0. \end{cases}$

Prove that f is differentiable at 0.

(b) Suppose that f is a function and c is a real number such that f is differentiable at c. Show that f is continuous at c.

(c) Let $g(x)=\begin{cases} 2x, & x\geq 0 \\ 0, & x\leq 0. \end{cases}$

Is g differentiable at 0? Justify your answer.

2. (a) Let f,g be functions and c be a real number such that $[c,c+\delta_0)\subseteq[\text{Domn}(f)\cap\text{Domn}(g)]$ and both f,g are right differentiable at c.

Prove that $f-g$ is right differentiable at c and $(f-g)'_{+}(c)=f'_{+}(c)-g'_{+}(c)$.

(b) Let $h(x)=\begin{cases} x^2-|x-2|, & x\geq 2 \\ 64-|x-2|, & x<2. \end{cases}$ Using the definition, show that h is right differentiable at $x=2$ and $h'_{+}(2)=3$.

Is h differentiable at 2? Justify your answer.

3. (a) State the chain rule for the derivative of a composition of two real-valued functions.

Let $f(x)=x^3+1$ and $g(x)=\begin{cases} x^2, & x\in\mathbb{Q}, \\ 0, & x\in\mathbb{Q}^c. \end{cases}$ Also, let $h(x)=f(g(x))$. Using the chain rule, prove that h is differentiable at 0 and $h'(0)=0$.

(b) Is it possible for two functions f and g with g not differentiable at a point $c \in \mathbb{R}$, $f \circ g$ is differentiable at $g(c)$? Justify your answer.

4. (a) Suppose that f is a function defined on an open interval (a,b) and suppose that f has a local minimum at $c \in (a,b)$. Prove that if f is differentiable at c, then $f'(c) = 0$.

(b) Let $g(x) = x^2 + 4|x|$ for all $x \in \mathbb{R}$. Show that $g(0)$ is the minimum of g, but $g'(0)$ does not exist. Why does this not contradict the result in (a) above.

5. (a) State Rolle's Theorem.

Let f be a three times differentiable function on $[a,b]$ and let $f(a) = f'(a) = f(b) = f'(b) = 0$. Show that there exists $c \in (a,b)$ such that $f'''(c) = 0$.

(b) Let $f(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ -1, & x \in \mathbb{Q}^c. \end{cases}$ Using Intermediate-Value property for derivatives, prove that there does not exist a function g defined on $\mathbb{R}$ such that $g'(x) = f(x)$ for each $x \in \mathbb{R}$.

6. (a) (i) Let I be an interval, x_0 be an interior point of I and let $n \geq 2$. Suppose that the derivatives $f^{(1)}, f^{(2)}, \ldots, f^{(n)}$ exists and are continuous in a neighborhood of x_0 and that $f^{(1)}(x_0) = f^{(2)}(x_0) = \cdots = f^{(n-1)}(x_0) = 0$, but $f^{(n)}(x_0) \neq 0$.

If n is even and $f^{(n)}(x_0) > 0$, prove that f has a relative minimum at x_0.

(ii) Show that $f(x) = x^4 + 1$, $x \in \mathbb{R}$ has a relative minimum at 0.

(b) Let $f(x) = \sqrt{1+x}$, $x \in [0, \infty)$.

(i) Find the 2nd order Taylor polynomial $P_2(x)$ for $f(x)$ at $x = 0$.

(ii) Prove that for each $x > 0$, $1 + \frac{x}{2} - \frac{x^2}{8} < \sqrt{1+x} < 1 + \frac{x}{2}$.