00063

**The Open University of Sri Lanka**
**B.Sc./ B.Ed. Degree Programme**
**Level-05 Final Examination-2013/2014**
**PUU3244/PUE5244 - Number Theory & Polynomials**
**Pure Mathematics**

**Duration: Three Hours.**

| **Date: 04.12.2014** | **Time: 1.30 p.m. - 4.30 p.m.** |
|---|---|

**Answer FIVE questions only.**

**(State clearly any results that you used to workout the Question 1 to Question 4, without proof.)**

1. (i) Define the set $\mathbb{N}$ of natural numbers.

   (ii) If $m, n \in \mathbb{N}$ then prove that $m+n \in \mathbb{N}$.

   (iii) If $x, y \in \mathbb{R}$ and $m, n \in \mathbb{N}$ prove that
   (a) $x^m.x^n = x^{m+n}$
   (b) $(x^m)^n = x^{mn}$
   (c) $(xy)^n = x^n y^n$

   (iv) Using mathematical induction, prove that

   $$1.5+2.5^2+3.5^3+\ldots n.5^n = \{(4n-1)5^{n+1}+5\}/16 \text{ for all positive integers } n.$$

2. (i) Define a prime number.

   (ii) If $x, y \in \mathbb{Z}$ and $3\mid(x^2+y^2)$ then prove that $3\mid x$ and $3\mid y$, where $\mathbb{Z}$ denotes the set of all integers.

   (iii) If $S$ is a non-empty subset of $\mathbb{Z}$ such that
   (a) $s_1, s_2 \in S \Rightarrow s_1 + s_2 \in S$. (closed under addition) and
   (b) $s_1, s_2 \in S \Rightarrow s_1 - s_2 \in S$. (closed under subtraction),

   then prove that $S$=0 or $S$ contains a least positive integer $d$ such that $S = \{nd : n \in \mathbb{Z}\}$.

   (iv) If $S$ is a subset of $\mathbb{Z}$ such that $S$ is closed under subtraction then prove that $S$ is closed under addition.

3. (i) Define each of the following for positive integers:
   (a) Greatest Common Divisor.
   (b) Least Common Multiple.
   (c) Pairwise relatively prime.

   (ii) If $a,\ b \in \mathbb{Z}$ (with at least one of them non-zero) then prove that $a$ and $b$ have a unique greatest common divisor $d$ which can be expressed in the form

   $d = am + bn$ with $m,\ n \in \mathbb{Z}$.

   (iii) Find the greatest common devisor of 4203 and 207. Express it in the form $4203m + 207n$ with suitable integers $m$ and $n$. Find the least common multiple of 4081 and 319.

4. (i) Show that for a given integer $n \in \mathbb{Z}$ there exist a unique $r \in \mathbb{Z}_m$ such that $n \equiv r (\text{mod } m)$ where $\mathbb{Z}_m = \{r \in \mathbb{Z} : 0 \le r < m\}$.

   (ii) Let $n \in \mathbb{N}$ and $p$ is a prime number. Prove that $n^p \equiv n \ (\text{mod } p)$.

   (iii) If $n \in \mathbb{N}$ prove that
   (a) $10^n \equiv 1 (\text{mod } 9)$. (b) $6^n \equiv 6 \ (\text{mod } 10)$.

   (iv) Prove that $2^{10} \equiv 1 (\text{mod } 31)$ and deduce that $2^{340} \equiv 1 (\text{mod } 31)$.

5. (i) Let $R$ be a commutative ring. If $f,\ g \in R[x]$ and $g$ is monic then prove that there exists unique $q,\ r \in R[x]$ such that $f = qg + r$ with $r = 0$ or $\deg(r) < \deg(g)$.

   (ii) Find the greatest common divisor of $f(x) = x^4 + 4x^3 + 3x^2 + x + 1$ and $g(x) = 2x^3 + x^2 - x - 3$ in $\mathbb{Z}_5[x]$ and express it in the form $d(x) = f(x)u(x) + g(x)v(x)$ where $d(x) = (f(x),\ g(x))$ and $u(x)$, $v(x)$ are functions in $\mathbb{Z}_5[x]$.

6. (i) State and prove Eisentein's irreducibility criteria.

   (ii) Determine whether the polynomial $25x^5 - 9x^4 - 3x^2 - 12$ in $\mathbb{Z}[x]$ is irreducible over $\mathbb{Q}[x]$.

   (iii) Express $f(x) = 2x^3 + 3x^2 - 7x - 5$ as a product of a unit and monic irreducible polynomials in $\mathbb{Z}_{11}[x]$.

7. (i) Let $f(x)=\sum_{i=0}^{n} a_i x^i \in \mathbb{Z}[x]$ and $n \geq 1$. If $\alpha \in \mathbb{Q}$ is a zero of $f(x)$ and $\alpha = \frac{r}{s}$ with $(r, s)=1$, then prove that $r \mid a_0$ and $s \mid a_n$.

(ii) Find all rational roots of the polynomial $64x^4 - 64x^3 - 4x^2 + 16x - 3$ over $\mathbb{Q}$.

8. (i) Let $f(x)=\sum_{i=0}^{n} a_i x^i \in \mathbb{C}[x]$, $a_n \neq 0$ and $\alpha_1, \alpha_2 \ldots\ldots \alpha_n$ are the zeros of $f(x)$ in $\mathbb{C}$.

Show that

(a) $a_n S_m + a_{n-1} S_{m-1} + \ldots\ldots + a_0 S_{m-n} = 0$ if $m > n$,

(b) $a_n S_m + a_{n-1} S_{m-1} + \ldots\ldots + a_{n-m+1} S_1 + m a_{n-m} = 0$ if $m \leq n$,

where $S_m = \sum_{i=0}^{n} \alpha_i^m$.

(ii) If $a, b, c \in \mathbb{C}$ such that $a+b+c=0$, then prove that,
$4(a^7+b^7+c^7) = 7abc(a^2+b^2+c^2)^2$.