The Open University of Sri Lanka
B.Sc. / B.Ed. Degree Programme
Pure Mathematics – Level 05
Final Examination - 2013/2014

PUU3240/PUE5240 - Ring Theory & Field Theory

Duration: Three Hours

Date: 12.11.2014 **Time: 09.30a.m. – 12.30p.m.**

Answer Five Questions Only.

1. (a) Let $(R;+,\cdot)$ be a ring and let $R\times\mathbb{Z}$ be a ring defined on binary operations $\oplus$ and Θ as follows:

$$(a,m)\oplus(b,n)=(a+b,m+n)$$
$$(a,m)\Theta(b,n)=(ab+mb+na,mn)$$

for $(a,m),(b,n)\in R\times\mathbb{Z}$.

(i) Find the additive identity and the additive inverse of $R\times\mathbb{Z}$.

(ii) Find the unity element of $R\times\mathbb{Z}$ if exists.

(iii) Prove that multiplication is associative on $R\times\mathbb{Z}$.

(b) Show that any ring R is commutative if it satisfies any one of the following conditions:

(i) $(a+b)^2=a^2+2ab+b^2$ for $a,b\in R$.

(ii) R is a ring with unity and $(ab)^2=a^2b^2$ for $a,b\in R$.

2. (a) Prove that the zero divisors of $\mathbb{Z}_n$ are precisely the non zero elements of $\mathbb{Z}_n$ which are not invertible.

(b) Show that $\mathbb{Z}\times\mathbb{Z}$ is not an integral domain.

3. (a) Prove that a commutative ring R with unity is a field if it has no proper ideals.

(b) Consider the subset $I=\left\{\begin{pmatrix} a & 0\\ b & 0\end{pmatrix} \mid a,b\in\mathbb{Z}\right\}$ of the ring $(M_2(\mathbb{Z});+,\cdot)$.

(i) Show that I is a subring of $M_2(\mathbb{Z})$.

(ii) Is I an ideal of $M_2(\mathbb{Z})$? justify your answer.

4. (a) (i) Let R and S be rings and let $\varphi : R \to S$ be a ring homomorphism. Suppose that e is an idempotent of R. prove that $\varphi(e)$ is an idempotent of S.

(ii) Suppose that s is any idempotent in S. define a map $\phi : \mathbb{Z} \to S$ by $\phi(n) = ns$ for all $n \in \mathbb{Z}$.

Show that ϕ is a ring homomorphism.

(b) Show that if ψ is a homomorphism defined on a ring R, then the set $\{a \in R \mid \psi(a) = a\}$ is a subring of R.

5. (a) State the Fundamental Homomorphism Theorem.

Let I and J be ideals of the ring R.

(i) Show that the mapping $\varphi : I \to (I+J)/J$ is an onto homomorphism.

(ii) Deduce that $I/(I \cap J) \cong (I+J)/J$.

(b) Prove that every homomorphic image of a nil ring is nil.

6. (a) Prove that a commutative ring with an identity is a field if and only if it has no non trivial ideals.

(b) Let f be a homomorphism from field F into a field F'. Show that f is the trivial homomorphism or else f is one to one.

7. (a) Let f be a homomorphism from the ring R into the ring R'.

In addition, R and R' are both rings with unity and $f(R) = R'$.

Prove each of the following:

(i) $f(-a) = -f(a)$ for all $a \in R$.

(ii) $f(1_R) = 1_{R'}$, where 1_R and $1_{R'}$ are identity elements of R and R'.

(iii) $f(a^{-1}) = [f(a)]^{-1}$ for each invertible element $a \in R$.

(b) For homomorphisms $\phi : (R;+,\cdot) \to (R';\oplus,\ominus)$ and $\psi : (R';\oplus,\ominus) \to (R'';\oplus',\ominus')$, show that their composition $\psi \circ \phi : (R;+,\cdot) \to (R'';\oplus',\ominus')$ is also a homomorphism.

8. Define the terms *prime ideal* and *maximal ideal* in a commutative ring R with unity.

(a) Let I be a proper ideal of the ring R.

Prove that I is a prime ideal if and only if the quotient ring R/I is an integral domain.

(b) Let R be a Boolean ring. Prove that non trivial ideal I of R is prime if and only if it is a maximal ideal.