

The Open University of Sri Lanka
 B.Sc. / B.Ed. Degree Programme
 Pure Mathematics – Level 05
 Final Examination - 2013/2014

PMU3295/PME5295 - Ring Theory

Duration: Three Hours



Date: 12.11.2014

Time: 09.30a.m. – 12.30p.m.

Answer Five Questions Only.

1. (a) Let $(R; +, \cdot)$ be a ring and let $R \times \mathbb{Z}$ be a ring defined on binary operations \oplus and \odot as follows:

$$(a, m) \oplus (b, n) = (a + b, m + n)$$

$$(a, m) \odot (b, n) = (ab + mb + na, mn)$$

for $(a, m), (b, n) \in R \times \mathbb{Z}$.

- (i) Find the additive identity and the additive inverse of $R \times \mathbb{Z}$.
- (ii) Find the unity element of $R \times \mathbb{Z}$ if exists.
- (iii) Prove that multiplication is associative on $R \times \mathbb{Z}$.

- (b) Show that any ring R is commutative if it satisfies any one of the following conditions:

(i) $(a + b)^2 = a^2 + 2ab + b^2$ for $a, b \in R$.

(ii) R is a ring with unity and $(ab)^2 = a^2b^2$ for $a, b \in R$.

2. (a) (i) Prove that the set $U = \{a \in R \mid ab = ba \text{ for each } b \in R\}$ in a given ring R is a commutative subring of R .

- (ii) Is the subset $S = \{n \mid n = 0 \text{ or } n \text{ is odd}\}$ of the ring $(R; +, \cdot)$ is a subring?

Justify your answer.

- (b) For given subrings U_1 and U_2 of a ring R , show that their intersection $U_1 \cap U_2$ is also a subring of R .

3. Let $R = R_1 \times R_2$ be the direct product of two rings R_1 and R_2 .

- (a) (i) Show that R has unit elements if R_1 and R_2 have unit elements.
- (ii) Show that R is commutative if R_1 and R_2 are commutative.

- (b) Is it true that R is an integral domain if R_1 and R_2 are integral domains? Justify your answer.

4. (a) (i) Let R and S be rings and let $\varphi: R \rightarrow S$ be a ring homomorphism. Suppose that e is an idempotent of R . prove that $\varphi(e)$ is an idempotent of S .
- (ii) Suppose that s is any idempotent in S . Define a map $\phi: \mathbb{Z} \rightarrow S$ by $\phi(n) = ns$ for all $n \in \mathbb{Z}$.
- Show that ϕ is a ring homomorphism.
- (b) Show that if ψ is a homomorphism defined on a ring R , then the set $\{a \in R \mid \psi(a) = a\}$ is a subring of R .
5. (a) State the Fundamental Homomorphism Theorem.
Let I and J be ideals of the ring R .
- (i) Show that the mapping $\varphi: I \rightarrow (I+J)/J$ is an onto homomorphism.
- (ii) Deduce that $I/(I \cap J) \cong (I+J)/J$.
- (b) Prove that every homomorphic image of a nil ring is nil.
6. (a) Prove that a commutative ring with an identity is a field if and only if it has no non trivial ideals.
- (b) Let f be a homomorphism from field F into a field F' . Show that f is the trivial homomorphism or else f is one to one.
7. (a) Let f be a homomorphism from the ring R into the ring R' .
In addition, R and R' are both rings with unity and $f(1_R) = 1_{R'}$.
Prove each of the following:
- (i) $f(-a) = -f(a)$ for all $a \in R$.
- (ii) $f(1_R) = 1_{R'}$, where 1_R and $1_{R'}$ are identity elements of R and R' .
- (iii) $f(a^{-1}) = [f(a)]^{-1}$ for each invertible element $a \in R$.
- (b) For homomorphisms $\phi: (R; +, \cdot) \rightarrow (R'; \oplus, \otimes)$ and $\psi: (R'; \oplus, \otimes) \rightarrow (R''; \oplus', \otimes')$,
show that their composition $\psi \circ \phi: (R; +, \cdot) \rightarrow (R''; \oplus', \otimes')$ is also a homomorphism.
8. Define the terms *prime ideal* and *maximal ideal* in a commutative ring R with unity.
- (a) Let I be a proper ideal of the ring R .
Prove that I is a prime ideal if and only if the quotient ring R/I is an integral domain.
- (b) Let R be a Boolean ring. Prove that non trivial ideal I of R is prime if and only if it is a maximal ideal.