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The Open University of Sri Lanka
B.Sc. / B.Ed. Degree Programme
Pure Mathematics – Level 04
Final Examination - 2013/2014

PUU2141/PUE4141- Continuous Functions

Duration: Two Hours

Date: 24.06.2014 **Time: 01.00p.m. – 03.00p.m.**

Answer Four Questions Only.

1. (a) Let $a, l \in \mathbb{R}$ and let f be a real-valued function.

   Define the statement that $\lim_{x \to a} f(x) = l$.

   (b) Let $f(x) = \begin{cases} 6, & \text{if } x < -2 \\ 5 - 3x, & \text{if } -2 \leq x \leq 1 \\ \dfrac{x-1}{\sqrt{x}+1}, & \text{if } x > 1. \end{cases}$

   (i) Show that the $\lim_{x \to 1^+} f(x) = 0$.

   (ii) Determine the existence of the limit of $f(x)$ of at the point 1.

   (c) Suppose that I is an interval, $c \in I$ or c is the right end point of I, $l \in \mathbb{R}$, and that $f : I \to \mathbb{R}$ is a function. Prove that if $\lim_{x \to c^-} f(x) = l$ there exists $\delta > 0$ and $M > 0$ such that for each $x \in (c - \delta, c)$, $|f(x)| \leq M$. Is its converse true? Justify your answer.

2. (a) Suppose $E \subseteq \mathbb{R}$, c is a limit point of E, $f, g : E \to \mathbb{R}$ are functions such that $\lim_{x \to c} f(x)$, $\lim_{x \to c} g(x)$ exist. Prove that $\lim_{x \to c} f(x)g(x)$ exists and

   $\lim_{x \to c} f(x)g(x) = \lim_{x \to c} f(x) \lim_{x \to c} g(x)$.

   (b) Let $f : (1,3) \to \mathbb{R}$ be defined by $f(x) = x^2\sqrt{x+7}$. Using part (a) show that $\lim_{x \to 2} f(x) = 12$.

   (c) If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} f(x)g(x)$ both exist, then does it follows that $\lim_{x \to a} g(x)$ exists?

   Justify your answer.

3. (a) State the definition of continuity of a function at a point.

   Let $f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ -x, & x \in \mathbb{Q}^c. \end{cases}$ Prove that f is continuous only at 0.

(b) Suppose that f is a function, a,b and c are real numbers such that $a<b<c$, $(a,c)\subseteq \text{Domn}(f)$, and that f is continuous at b. Prove that f is left-continuous at b.

(c) Determine the value of the constant A so that the following function is continuous for all values of $x\in\mathbb{R}$:

$$f(x)=\begin{cases} Ax^2-A, & \text{if } x\geq 3 \\ 4, & \text{if } x<3. \end{cases}$$

4. (a) (i) State the sandwich theorem for limits of functions at infinity.

(ii) Show that $\lim_{x\to\infty}\left(\dfrac{x^2\left(2+\sin^2(x)\right)}{x+100}\right)=\infty$.

(b) Let f be a function defined on $\mathbb{R}$, and let c and k be real numbers. Prove that if $\lim_{x\to c^-}=f(x)=\infty$, then for each $k\in\mathbb{R}$, $\lim_{x\to c^-}\left(f(x)+k\right)=\infty$.

Deduce that $\lim_{x\to 0^-}\left(\dfrac{1}{x^2}-1\right)=\infty$.

5. (a) Let I be an interval , and f be a function such that f is uniformly continuous on I. Prove that f is continuous on I.

(b) If a function $f:[0,\infty)\to\mathbb{R}$ is continuous on $[0,\infty)$ and uniformly continuous on $[k,\infty)$ for some $k>0$, show that f is uniformly continuous on $[0,\infty)$. Let $f:[0,\infty)\to[0,\infty)$ be defined by $f(x)=\sqrt{x}$. Deduce that f is uniformly continuous on $[0,\infty)$.

6. (a) Let a,b and c be real numbers, and f and g be functions such that $\lim_{x\to a}f(x)=b$ and $\lim_{x\to b}g(x)=c$. Also, let g be continuous at b. Prove that $\lim_{x\to a}g(f(x))=c$.

Let f and g be the functions given by $f(x)=\begin{cases} 2, & x\in\mathbb{R}\setminus\{1\} \\ 10, & x=1. \end{cases}$ and

$$g(x)=\begin{cases} x^2-2x+4, & x\geq\dfrac{19}{10} \\ \dfrac{1}{x-\left(\dfrac{19}{10}\right)}, & x<\dfrac{19}{10}. \end{cases}$$

Deduce that $\lim_{x\to 1}g(f(x))=3$.

(b) Find a function $f:\mathbb{R}\to\mathbb{R}$ such that for each $c\in\mathbb{R}$, $\lim_{x\to c}f(x)$ does not exist and $\lim_{x\to c}f(f(x))$ exists.