The Open University of Sri Lanka
B.Sc./B.Ed. Degree Programme
Pure Mathematics – Level 04
Final Examination-2013/2014

PUU2140/PUE4140- Sequences and Series

Duration: Two Hours

Date: 07.06.2014 **Time: 01.00 p.m. to 03.00 p.m.**

Answer Four Questions Only.

1. (a) Let $\langle x_n \rangle$ be a sequence of real numbers and let $l \in \mathbb{R}$.
State the $\varepsilon - N$ definition for $\lim_{n\to\infty} x_n = l.$

Using the definition show that $\lim_{n\to\infty}\left(\frac{n}{\sqrt{n^2+n}}\right) = 1.$

Deduce that the sequence $\left\langle \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}} \right\rangle$ converges.

(b) Prove that every convergent sequence of real numbers is bounded.
Show that if $\langle x_n \rangle$ is a bounded sequence of positive numbers, then

$$\lim_{n\to\infty} \frac{x_n}{x_1 + x_2 + \cdots + x_n} = 0.$$

2. (a) Prove that every convergent sequence of real numbers is a Cauchy sequence.

Let $\langle a_n \rangle$ be the sequence defined by $a_n = (-1)^n\left(\frac{n+1}{n}\right)$ for each $n \in \mathbb{N}$.
Show that $\langle a_n \rangle$ is not a Cauchy sequence.
Is it convergent? Justify your answer.

(b) Is it true that $\left\langle \frac{x_n}{y_n} \right\rangle$ is Cauchy, where $\langle x_n \rangle$ and $\langle y_n \rangle$ are two Cauchy sequences such that $y_n \neq 0$ for each $n \in \mathbb{N}$? Justify your answer.

3. (a) Using the definition show that the sequence $\langle 4 + (-1)^n \rangle$ diverges.

(b) Let $\langle x_n \rangle$ be a bounded divergent sequence of real numbers and let $\langle y_n \rangle$ be a sequence that converges to zero. Prove that the sequence $\langle x_n y_n \rangle$ converges to zero.

Deduce that $\left\langle -\frac{1+(-1)^n}{2n} \right\rangle$ converges to zero.

4. Prove or disprove each of the following:

(a) If $\langle x_n \rangle$ and $\langle y_n \rangle$ are two sequences such that $\langle x_n \rangle$ is a bounded divergent sequence and $\langle y_n \rangle$ is a convergent sequence then $\langle x_n + y_n \rangle$ is a bounded divergent sequence.

(b) Let $\langle a_n \rangle$ and $\langle b_n \rangle$ be sequences of real numbers such that $\lim_n a_n = \infty$, $\lim_n b_n = -\infty$. Then $\lim_n a_n b_n = -\infty$.

(c) Every oscillating sequence is bounded.

(d) At least one tail of $\langle x_n \rangle$ is Cauchy, then the sequence $\langle x_n \rangle$ is Cauchy.

(e) If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are positive-term series and $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} (a_n + b_n)$ diverges.

5. (a) Prove that whenever the root test is inconclusive with regard to the convergence of a series of real numbers, the ratio test is also inconclusive with regard to the convergence of it.

(b) Stating clearly the tests you use, test each of the following series for convergence:

(i) $\sum_{n=1}^{\infty} \frac{n!}{e^n}$ (ii) $\sum_{n=1}^{\infty} \frac{2+\cos n}{3^n}$ (iii) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n-1}}$

(iv) $\sum_{n=1}^{\infty} \frac{2^n}{2^n+3^n}$ (v) $\sum_{k=1}^{\infty} \frac{k^{20}}{k!}$

(c) Find the values of x for which the series $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges.

6. (a) Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series. Prove that $\sum_{n=1}^{\infty} (a_n - b_n)$ is a convergent series and $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$.

(b) Prove that the absolute convergence implies convergence of a series using the following steps:

(i) Show that $0 \le x + |x| \le 2|x|$ for every real number x.

(ii) Use the Part (i) to show that if $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} (a_n + |a_n|)$ converges.

(iii) Use the part (ii) to show that $\sum_{n=1}^{\infty} a_n$ converges, if $\sum_{n=1}^{\infty} |a_n|$ converges.