The Open University of Sri Lanka
B.Sc./B.Ed Degree Programme – Level 04
Final Examination 2013/2014
Pure Mathematics
PMU 2191/PME 4191– Vector Analysis.

Duration :- Two Hours.

Date :- 14. 06. 2014 **Time:- 1.30 p.m. – 3.30 p.m.**

Answer Four Questions Only.

1. (a) If $u = f(x,y)$, where $x = e^s \cos t$ and $y = e^s \sin t$, t being a parameter, show that

(i) $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = e^{-2s}\left[\left(\frac{\partial u}{\partial s}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2\right]$ and

(ii) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-2s}\left[\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2}\right]$.

(b) Find the maximum and minimum values of the function $f(x, y) = x^2 + y^2 + x^2 y + 4$ and determine their nature.

2. (a) Prove that grad ϕ is a vector normal to the contour surface $\phi(x,y,z) = c$, where c is a constant.

(b) (i) Show that the equation of the tangent plane to the surface $F(x, y, z) = 0$ at the point $P(x_0, y_0, z_0)$ is given by

$$\frac{x - x_0}{\left(\frac{\partial F}{\partial x}\right)_P} = \frac{y - y_0}{\left(\frac{\partial F}{\partial y}\right)_P} = \frac{z - z_0}{\left(\frac{\partial F}{\partial z}\right)_P}.$$

(ii) Show that the equation of the tangent plane to the elliptic parabaloid $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ at the point $P(x_0, y_0, z_0)$ can be written as $\frac{z + z_0}{c} = \frac{2xx_0}{a^2} + \frac{2yy_0}{b^2}$.

(c) Suppose that over certain region of space the electric potential $V(x, y, z)$ is given by $V(x, y, z) = 5x^2 - 3xy + xyz$.

(i) Find the rate of change of the potential at $P(3, 4, 5)$ in the direction of the vector $\underline{v} = \underline{i} + \underline{j} + \underline{k}$.

(ii) In which direction does $V(x, y, z)$ change most rapidly.

(iii) What is the maximum rate of change at P

3. (a) Find the surface integral of the function $f(x,y)=x+2y$ over the region bounded by $y=2x^2$ and $y=1+x^2$.

(b) Use plane polar coordinates to evaluate the surface integral of the function $f(x,y)=3x+4y^2$ over the region in the upper half-plane bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.

(c) Using cylinderical polar coordinates find the volume integral of the function $f(x,y,z)=x^2+y^2$ over the closed region bounded by $z=\sqrt{x^2+y^2}$ and $z=2$.

4. (a) State Gauss' Divergence theorem.

(b) Verify the above theorem considering the vector field $\underline{F}=3x\underline{i}+xy\underline{j}+2xz\underline{k}$ and closed region bounded by the planes $x=0,\ x=1,\ y=0,\ y=1,\ z=0$ and $z=1$.

(c) Let $\underline{r}=x\,\underline{i}+y\,\underline{j}+z\,\underline{k}$ and $r=|\underline{r}|$ then prove that

(i) $\underline{\nabla}\cdot\underline{r}=3$, (ii) $\underline{\nabla}\cdot r\underline{r}=4r$, (iii) $\underline{\nabla}^2 r^3=12r$

5. (a) (i) State Stokes' Theorem.

(ii) Verify Stokes' Theorem considering the vector field $\underline{F}=y\underline{i}+z\underline{j}+x\underline{k}$, and S the hemisphere $x^2+y^2+z^2=1,\ y\geq 0$, oriented to the direction of the positive y-axis.

(b) Show that $\underline{F}=(1+xy)e^{xy}\underline{i}+\left(e^y+x^2e^{xy}\right)\underline{j}$ is a conservative vector field. Then find a scalar function ϕ such that $\underline{F}=\underline{\nabla}\phi$.

6. (a) Suppose that S is a plane surface lying in the xy–plane bounded by a closed curve C. If $\underline{F}=P(x,y)\underline{i}+Q(x,y)\underline{j}$ then show that $\oint_C(Pdx+Qdy)=\iint_S\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy$.

(b) Verify the above result for the integral $\oint_C xy^2dx-x^2ydy$, where C is the circle given by $x^2+y^2=4$, with counterclockwise direction.