The Open University of Sri Lanka
B.Sc./B.Ed. Degree Programme
Final Examination-2013/2014
APU3143/APE5143-Mathematical Methods
Applied Mathematics -Level 05

Duration: Two Hours.

Date: 03.07.2014	**Time: 1.00 p.m.- 3.00p.m.**

Answer FOUR questions only.

1. The Laplace transform of a function $f(t)$, denoted by $L[f(t)]$ is defined as

$$L[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}dt \quad \text{and} \quad L^{-1}\{F(s) = f(t)\}.$$

(a) Find the inverse Laplace transforms of the following function.

$$F(s) = \frac{3s+1}{(s+1)^4}.$$

(b) Show that $L^{-1}\left\{\dfrac{2+5s}{s^2e^{4s}}\right\} = \begin{cases} 0 & 0 < t < 4 \\ 2t-3 & t > 4. \end{cases}$

(c) Use the convolution theorem to find the inverse Laplace transform of

$$\frac{1}{(s+1)(s+9)^2}.$$

2. Solve each of the following boundary value problems using the Laplace transform method:

(a) $\dfrac{d^2y}{dt^2} + 16y = 32t$, subject to $y(0) = 3,\ y'(0) = -2.$

(b) $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 4y = 6e^{-2t}$, subject to $y(0) = -2,\ y'(0) = 8.$

(c) $\dfrac{d^3y}{dt^3} + 8y = 32t^3 - 16t$, subject to $y(0) = y'(0) = y''(0) = 0.$

45

3. Obtain the formal expansion of the function f defined by $f(x)=x \ (0 \le x \le \pi)$ as a series of orthonormal characteristic functions $\{\phi_n\}$ of the Sturm-Liouville boundary value problem

$$\frac{d^2y}{dx^2}+\lambda y=0$$
$$y(0)=0$$
$$y(\pi)=0.$$

4. (a) Consider the function $f(x)$ defined by

$$f(x)=\begin{cases}0, & -\pi \le x < 0\\ 2, & 0 \le x \le \pi\end{cases}$$

Find the trigonometric Fourier series of $f(x)$ in $-\pi \le x \le \pi$.

(b) Consider the function $f(x)$ defined by $f(x)=x; \quad 0 \le x \le \pi$.

Find the Fourier sine series and the Fourier cosine series of $f(x)$ on $0 \le x \le \pi$.

5. (a) The Gamma function, denoted by $\Gamma(p)$ corresponding to the parameter p, is defined by the improper integral $\Gamma(p)=\int_0^\infty e^{-t}t^{p-1}dt, \quad (p>0)$.

Evaluate each of the following integrals using Gamma functions:

(i) $\int_0^\infty \sqrt{ye^{-y^2}}\,dy$ (ii) $\int_0^1 \frac{dx}{\sqrt{-\ln x}}$.

(b) The Beta function denoted by $\beta(p,q)$ is defined by $\beta(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}\,dx,$

where $p>0$ and $q>0$ are parameters.

Prove each of the following results using Beta functions:

(i) $\int_0^{\frac{\pi}{2}}\left(\sqrt{\tan\theta}+\sqrt{\sec\theta}\right)d\theta=\frac{1}{2}\Gamma\left(\frac{1}{4}\right)\left\{\Gamma\left(\frac{3}{4}\right)+\frac{\sqrt{\pi}}{\Gamma\left(\frac{3}{4}\right)}\right\}$.

(ii) $\int_0^1 x^4 \left[\ln\left(\frac{1}{x}\right)\right]^3 dx = \frac{6}{625}$.

6. $J_p(x)$ the Bessel function of order p is given by the expansion

$$J_p(x) = x^p \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+p}.m!\Gamma(p+m+1)}.$$

(a) Prove the following recurrence relation:

$$\frac{d}{dx}\{J_p(x)\} = J_{p-1}(x) - \frac{p}{x}J_p(x) \text{ or } xJ_p'(x) = xJ_{p-1}(x) - pJ_p(x)$$

$$\left(\text{Hint: } \frac{d}{dx}\{x^p J_p(x)\} = x^p J_{p-1}(x)\right)$$

(b) Evaluate $J_{-\frac{1}{2}}$, $J_{\frac{3}{2}}$, in terms of $\sin x$ and $\cos x$.

$$\left(\text{Hint: } J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}.\sin x.\right)$$

(c) Evaluate $\int J_3(x)dx$.

46