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The Open University of Sri Lanka
B.Sc./ B.Ed. Degree Programme
Final Examination-2014/2015
Pure Mathematics- Level-05
PUU 3244 - Number Theory & Polynomials

Duration: Three Hours.

Date: 11-11-2015 **Time: 1.30p.m. To 4.30p.m.**

Answer Five questions only.

(State clearly any results that you used, without proof.)

1. (a) Prove each of the following:

   (i) If $n \in \mathbb{N}$ then $n \geq 1$

   (ii) If $n \in \mathbb{N}$ then there is no $m \in \mathbb{N}$ such that $n < m < n+1$

   (b) Show that the equation $n + x = m$ with $m,\ n \in \mathbb{N}$ has a solution in $\mathbb{N}$ if and only if $m > n$.

   (c) Show that the equation $n + 10 = 7$ has no solution in $\mathbb{N}$.

2. (a) State three forms of Mathematical Induction.

   (b) Using mathematical induction, prove that $5^{n+1} - 2^{n+1} - 3^{n+1}$ is divisible by 6 for all positive integers n.

   (c) Prove each of the following:

   (i) $\mathbb{Z} = \mathbb{N} \cup (-\mathbb{N}) \cup \{0\}$

   (ii) $\mathbb{Z}$ is not a well-ordered set.

3. (a) If $n \in \mathbb{N}$ and $n > 1$ then prove that there exist a prime number p such that $p \mid n$.

   (b) Show that $(x^n + y^n)$ is divisible by $(x + y)$ when n is an odd positive integer.

   (c) If x and y are odd positive integers then show that $(x^2 + y^2)$ is even but not divisible by

(d) (i) Find the greatest common divisor g of 35077 and 2821, and then find integers x and y which satisfy $35077x + 2821y = g$.

(ii) Find the least common multiple of 35077 and 2821.

4. (a) Prove each of the following:

(i) If $a + c \equiv b + c \pmod{m}$ then

$a \equiv b \pmod{m}$

(ii) If $ac \equiv bc \pmod{m}$ and $(c,m)=1$ then

$a \equiv b \pmod{m}$

(iii) If $ac \equiv bc \pmod{m}$ and $(c,m)= d$ then

$a \equiv b \pmod{\frac{m}{d}}$.

(5) (i) Let R be a commutative ring. If $f(x), g(x) \in R[x]$ and $g(x)$ is monic then prove that there exists a unique $q(x), r(x) \in R[x]$ such that $f(x) = q(x)g(x) + r(x)$ with $r(x) = 0$ or $\deg(r(x)) < \deg(g(x))$.

(ii) If $f(x) = x^4 + 2x^3 + 2x^2 + 2x + 1$ and $g(x) = x^2 - 1$ are polynomials over $\mathbb{Q}[x]$. Find the greatest common divisor $d(x)$ of $f(x)$ and $g(x)$ and express it in the form $d(x) = f(x)u(x) + g(x)v(x)$ with $u(x), v(x) \in \mathbb{Q}[x]$.

(6) (i) State and prove, Eisentein's irreducibility criteria.

(ii) Determine whether the polynomial $x^4 + x^3 + x^2 + x + 1$ is irreducible over $\mathbb{Q}[x]$ or not.

(7) (i) Let $f(x) = \sum_{i=0}^{n} a_i x^i \in \mathbb{Z}[x]$ and $n \geq 1$. If $\alpha \in \mathbb{Q}$ is a zero of $f(x)$ and $\alpha = \frac{r}{s}$ with $(r, s) = 1$, then show that $r \mid a_0$ and $s \mid a_n$.

(ii) Find all rational roots of the polynomial $20x^3 - 47x^2 + 36x - 9$ over $\mathbb{Q}$.

(iii) State and prove Remainder Theorem.

(iv) Find all prime numbers p such that $x-3$ is a factor of $f(x)=x^4+x^3+x^2+x+1$ in $\mathbb{Z}_p[x]$.

(8) (i) Let $f(x)=\sum_{i=0}^{n} a_i x^i \in \mathbb{C}[x]$, $a_n \neq 0$ and $\alpha_1, \alpha_2 \ldots\ldots \alpha_n$ are the zeros of $f(x)$ in $\mathbb{C}$.

Show that

(a) $a_n S_m + a_{n-1} S_{m-1} + \ldots\ldots + a_0 S_{m-n} = 0$ if $m > n$,

(b) $a_n S_m + a_{n-1} S_{m-1} + \ldots\ldots + a_{n-m+1} S_1 + m a_{n-m} = 0$ if $m \leq n$,

where $S_r = \sum_{i=0}^{n} \alpha_i^r$.

(ii) If $a, b, c \in \mathbb{C}$ such that $a+b+c=0$, then prove that,

$$4(a^7+b^7+c^7) = 7abc(a^2+b^2+c^2)^2.$$