The Open University of Sri Lanka

B.Sc./B.Ed. Degree Programme

Final Examination-2015/2016

APU3143/APE5143-Mathematical Methods

Applied Mathematics -Level 05

Duration: Two Hours.

Date: 25.07.2016	**Time: 1.00 p.m.- 3.00 p.m.**

Answer FOUR questions only.

01. The Laplace transform of a function *f(t)*, denoted by *L[f(t)]* is defined as

$$L[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}dt \text{ and the inverse transform as } L^{-1}\{F(s)\} = f(t).$$

(a) If $L^{-1}\left\{\frac{\omega s}{(s^2+\omega^2)^2}\right\} = \frac{t\sin t}{2}$, show that $L^{-1}\left\{\frac{8s}{(4s^2+1)^2}\right\} = \frac{t}{2}\sin\frac{t}{2}$.

(b) Find the inverse Laplace transforms of each of the following functions:

(i) $F(s) = \frac{1}{2}\ln\frac{s^2+b^2}{s^2+a^2}$. (ii) $F(s) = \left\{\frac{s}{(s-2)^2(s+1)}\right\}$.

(c) Show that $L^{-1}\left\{\frac{e^{-s}}{(s+1)^3}\right\} = \frac{1}{2}e^{-(t-1)}(t-1)^2u(t-1)$;

where $u(t)$ is the unit step function.

02. (a) Use the convolution theorem to find the inverse Laplace transform of

$$\frac{s^2}{(s^2+a^2)(s^2+b^2)}.$$

(b) Solve each of the following boundary value problems using the Laplace transform method:

(i) $\frac{d^2y}{dt^2} + 4y = 8$, subject to $y(0)=0, y'(0)=6$.

(ii) $\dfrac{d^2y}{dt^2}+2\dfrac{dy}{dt}+5y=0$ subject to $y(0)=2, y'(0)=4$.

03. What is the Sturm-Liouville equation?

(a) A boundary value problem is given by

$$\frac{d^2y}{dx^2}+\lambda y=0$$
$$y(0)=0$$
$$y(\pi)=0$$

Is it a Sturm-Liouville problem? Justify your answer.

(b) Find non-trivial solutions of the problem

$$\frac{d^2y}{dx^2}+\lambda y=0$$
$$y(0)=0, y(\pi)=0$$

(c) Let $\{\phi_n\}$ be the orthogonal characteristic functions of the problem in part (a). Form the sequence of orthonormal characteristic functions $\{k_n\phi_n\}$

where $k_n=\dfrac{1}{\sqrt{K_n}}=\dfrac{1}{\sqrt{\int_a^b[\phi_n(x)]^2\,r(x)\,dx}}$ $\qquad (n=1, 2, 3\ldots)$

4. (a) Let $f(x)$ be a function defined in the interval $-\pi<x<\pi$. The Fourier series of $f(x)$ is given by

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos nx+b_n\sin nx \text{ where}$$

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)dx\ ,\ a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx \text{ and } b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx.$$

Show that $\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}[f(x)]^2\,dx=2a_0^2+\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)$.

(b) Consider the function $f(x)$ defined by

$$f(x)=\begin{cases}\dfrac{2}{\pi}x+2 & \text{when } -\pi<x<0\\ 2 & \text{when } 0\le x<\pi\end{cases}$$

Find the trigonometric Fourier series of $f(x)$ in $-\pi\le x\le\pi$.

05. (a) The Gamma function, corresponding to the parameter p denoted by $\Gamma(p)$ is defined by the improper integral

$$\Gamma(p) = \int_0^{\infty} e^{-t}\, t^{p-1} dt, \quad (p > 0).$$

(i) Evaluate the following integral using Gamma function:

$$\int_0^1 \sqrt[3]{x \ln\left(\frac{1}{x}\right)}\, dx$$

(ii) Show that the area under the normal curve $y = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$ and x-axis is unity,

where σ is a positive constant.

(b)The Beta function denoted by $\beta(p,q)$ is defined by $\int_0^1 x^{p-1}(1-x)^{q-1} dx$,

where $p > 0$ and $q > 0$ are parameters.

Evaluate each of the following intrigals using Beta function:

(i) $\displaystyle\int_0^1 \left(\frac{x^3}{1-x^3}\right)^{\frac{1}{2}} dx.$ (ii) $\displaystyle\int_0^{\infty} \frac{x^2}{1+x^4} dx.$

06. Let $J_p(x)$ be the Bessel function of order p given by the expansion

$$J_p(x) = x^p \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+p}.m!\Gamma m!+ m+1)}$$

(a) Show that $J_2'(x) = \left(1-\dfrac{4}{x^2}\right) J_1(x) + \dfrac{2}{x} J_0(x)$.

(b) Express $J_{\frac{7}{2}}(x)$ in terms of sine and cosine functions.

(c) Evaluate $\int x^4 J_1(x) dx$.