The Open University of Sri Lanka
B.Sc./ B.Ed. Degree Programme
Final Examination-2015/2016
Pure Mathematics-Level 5
PMU3291/PME5291 - Complex Analysis
Duration: Three Hours

Date: 07.07.2016 **Time: 9.30a.m.-12.30p.m.**

Answer Five Questions Only.

1. (a) Let $z_n = e^{in\theta}$, where $\theta \in \mathbb{R}$ and $0 \le \theta < \pi$ Show that $\lim_{n\to\infty} z_n$ exist only when $\theta = 0$.

(b) Show that $(z^{10} - 1) = (z^2 - 1)\prod_{k=1}^{4}(z^2 - \alpha_k z + 1)$, where $\alpha_k = 2\cos\left(\frac{k\pi}{5}\right)$ for $k = 1, 2, 3, 4$.

2. Let $f(z)$ be analytic in a region G. Show that

(i) if $f'(z)=0$ in G, then $f(z)$ is constant in G,
(ii) if $\text{Im}f(z)$ is constant in G, then $f(z)$ is constant in G,
(iii) if $|f(z)|$ is constant in G, then $f(z)$ is constant in G.

3. (a) Let $(1 - z - z^2)^{-1} = \sum_{n=0}^{\infty} F_n z^n$. Prove that $F_0 = F_1 = 1,\ F_n = F_{n-1} + F_{n-2}$.

Show that $F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$.

(b) Let $(z) = \frac{1}{(z-1)(z-2)}$. Find the Laurent series expansion of $f(z)$ in each of the following annuli:
(i) $1 < |z| < 2$, (ii) $|z| > 2$, (iii) $0 < |z - 1| < 1$.

4. (a) State the Maximum Modulus Theorem.

(b) Find the maximum value of $|z^2 + 3z - 1|$ in the closed disk $\{z | z \in \mathbb{C}, |z| \leq 1\}$.

5. (a) State Rouche's Theorem.

(b) Prove that all the roots of $z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$.

6. (a) Find the singularities of the function $\dfrac{1}{\sin\frac{\pi}{z}\cos\frac{\pi}{z}}$. Classify each of these as isolated singularities or non-isolated singularities.

(b) For each of the following functions find the isolated singularities. In each of the isolated singularities find the Laurent series expansion around that point. Hence classify those singularities.

(i) $\dfrac{\cos z - 1}{z}$ (ii) $\dfrac{z+i}{z^2+1}$ (iii) $e^{-\frac{1}{z}}$.

7. (a) Evaluate $\displaystyle\int_C \frac{1}{z^2(z^2+2z+2)}dz$, where C is the circle with radius 3, centered at 0 oriented counterclockwise.

(b) Use the method of residues to show that $\displaystyle\int_0^{2\pi} \frac{1}{5+4\sin\theta}d\theta = \frac{2\pi}{3}$.

8. Use the method of residues to show that,

(i) $\displaystyle\int_0^{\infty} \frac{1}{(x^2+a^2)(x^2+b^2)}dx = \frac{\pi}{2ab(a+b)}$, where $a, b > 0$.

(ii) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^{n+1}}dx = \frac{(2n)!\pi}{2^{2n}(n!)^2}$, where n is a positive integer.