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The Open University of Sri Lanka
B.Sc. Degree Programme
Level 05- Final Examination 2015/2016
Pure Mathematics
PMU 3292/PME 5292- Group Theory and Transformation

Duration :- 3 Hours

Date :- 13-07-2016 **Time:- 1.00 pm- 4.30 pm**

Answer FIVE questions only.

1.

(a) Prove that the set of the six special bilinear mappings $f_1, f_2, f_3, f_4, f_5, f_6$ defined by $f_1(z) = z, f_2(z) = \frac{1}{z}, f_3(z) = 1 - z, f_4(z) = \frac{z}{z-1}, f_5(z) = \frac{1}{1-z}, f_6(z) = \frac{z-1}{z}$ of the infinite complex plane into itself is a finite non-abelian group of order 6.

(b) If $(G, °)$ is group, then prove that the followings:

(i) The identity element of G is unique.

(ii) Every $a \in G$ has a unique inverse, $a^{-1} \in G$.

2.

(a) Show that set $S = \{I, \ (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$ form a group under the permutation multiplication, where I is the identity element of S.

(b) Prove that the set of nonzero matrices $\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$ where $x, y \in \mathbb{R}$ form a group under the matrix multiplication.

3.

(a) Which of the following subsets of S_4 are subgroups? Justify your answer.

(i) $\{I, (1\ 2\ 3), (1\ 3\ 2\ )\}$

(ii) $\{I, (1\ 2\ 3), (1\ 3\ 2\ ), (1\ 3\ 4\ ), (1\ 4\ 3)\}$

(iii) $\{I, (1\ 2\ 3\ 4\ ), (1\ 3\ )(2\ 4), (1\ 4\ 3\ 2)\}$

(b) Prove that a non empty subset H of a group G is a subgroup of G if for all $x, y \in H$, the element $xy^{-1} \in H$.

4.

(a) Determine the elements of the cyclic group generated by the matrix $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ under the matrix multiplication.

(b) Let a and b be two elements of a group G. Assume that a has order 5 and that $a^3b = ba^3$. Prove that $ab = ba$.

(c) What is the subgroup of $(\mathbb{Z}_{18}, \oplus)$ generated by $\bar{3}$?

5.

(a) Let H be a subgroup of G. Prove that $|Ha| = |aH| = |H| \ \forall\, a \in G$, where Ha and aH are right and left cosets of H respectively.

(b) Let H be the cyclic subgroup (of order 2) of S_3 generated by $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$. Prove that no left coset of H(except H itself) is also a right coset of H.

6.

(a) Let H and K be subgroups of a group G. Prove that $H \cap K$ is also a subgroup of G. Is $H \cup K$ a subgroup of G? Justify your answer.

(b) Show that $H = \{5x |\ x \in \mathbb{Z}\}$ is a subgroup of $(\mathbb{Z}, +)$.

(c) Write down all the distinct right cosets of $H = \{5x |\ x \in \mathbb{Z}\}$.

7.

(a) Prove that,N is a normal subgroup of G if and only if $gNg^{-1} = N, \forall\, g \in G$.

(b) Let $G = (\mathbb{Z}, +)$ and $N = \{2x |\ x \in G\}$ .Show that $N \lhd G$.

(c) If H is a subgroup of a group G and $N = \{\, g \in G |\ gHg^{-1} = H\}$. Prove that $H \lhd N$.

8.

(a) Let ϕ be a homomorphism of G into $\bar{G}$. Prove the followings:

(i) $\phi(e) = \bar{e}$,where e and $\bar{e}$ are unit elements of G and $\bar{G}$ respectively.

(ii) $\phi(x^{-1}) = [\phi(x)]^{-1};\ \forall\, x \in G$.

(b) Let $G = (\mathbb{R}, +)$, $\bar{G} = (\mathbb{R} \setminus \{0\}, .)$. Define $\phi : G \to \bar{G}$ by $\phi(x) = 2^x$. Show that ϕ is a homomorphism and determine $ker\phi$.