The Open University of Sri Lanka
B.Sc/B.Ed. Degree Programme
Final Examination - 2015/2016
Pure Mathematics - Level 05
PUU3141/PUE5141 Algebra of Complex Numbers

Duration: Two Hours

Date: 22.07.2016 Time: 09.30am - 11.30am

Answer FOUR questions ONLY.

NOTE: Throughout this paper $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{Z}$ denotes the set of integers.

1. (a) Let $z \in \mathbb{C}$ such that $Imz > 0$. Prove that $Im(z^{-1}) < 0$ and $Im((\overline{z})^{-1}) > 0$.

   (b) Prove that $z^2 \in \mathbb{R}$ if and only if z is pure imaginary or $z \in \mathbb{R}$.

   (c) Find $z \in \mathbb{C}$ such that

      i. $Re(z(1+i)) + z\overline{z} = 0$,

      ii. $Re(z^2) + iIm(\overline{z}(1+2i)) = -3$.

2. (a) Let $z \in \mathbb{C}$. Suppose that $|z| = \frac{1}{2}$. Prove that

      i. $|z^2 - 3z + 5| \leq \frac{27}{4}$,

      ii. $|z^5 - 6| \geq \frac{191}{32}$.

   (b) Let $z \in \mathbb{C}$ and $a \in \mathbb{R}$ such that $z \neq a$ and $|z| = a$. Show that $i\left(\frac{a+z}{a-z}\right) \in \mathbb{R}$.

   (c) Let $z \in \mathbb{C}$ such that $|z - 3 - i| = |z - 4 - 2i|$ and $|z - 5 - 7i| = |z + 1 + i|$. Find z.

3. (a) Find all the complex numbers z such that $arg(z+1) = \frac{\pi}{4} + 2n\pi$, $n \in \mathbb{Z}$ and $arg(z-1) = \frac{3\pi}{4} + 2n\pi$, $n \in \mathbb{Z}$.

   (b) Suppose $z_1, z_2 \in \mathbb{C}$ and $z_1 \neq z_2$ such that $|z_1 + \overline{z}_2| = |z_1 - \overline{z}_2|$. Show that $arg(z_1) + arg(z_2) = \frac{\pi}{2} + 2n\pi$, $n \in \mathbb{Z}$ or $arg(z_1) + arg(z_2) = -\frac{\pi}{2} + 2n\pi$, $n \in \mathbb{Z}$.

(c) Let $z \in \mathbb{C}$. Find the locus of $Re\left(\frac{z-4i}{z+2i}\right) = 0$.

4. State the De Moivre's theorem for any $n \in \mathbb{Z}$.

   (a) Write the complex number $(1+i)(\sqrt{3}+i)$ in polar form and hence deduce the values for $\cos\left(\frac{5\pi}{12}\right)$ and $\sin\left(\frac{5\pi}{12}\right)$.

   (b) Evaluate $\left[(\sqrt{3}-1)+i(\sqrt{3}+1)\right]^{-24}$.

   (c) Evaluate $(-81)^{\frac{1}{4}}$. Deduce the list of complex numbers for $(-81)^{\frac{3}{4}}$.

5. (a) Let $z \in \mathbb{C}$. Prove that $|e^z| = 1$ if and only if z is pure imaginary.

   (b) Show that $Re\left[\frac{1}{1-(e^{i\theta}/2)}\right] = \frac{4-2\cos\theta}{5-4\cos\theta}$ for any $\theta \in \mathbb{R}$.

   (c) Let $z \in \mathbb{C}$. Solve $\frac{1}{2}\sin 2z - \sin z + 2i(1-\cos z) = 0$.

6. Let $z \in \mathbb{C}$.

   (a) Show that the complex number $2+3i$ is a root of the equation $z^4 - 12z^3 + 62z^2 - 172z + 221 = 0$.

   Hence find all the roots of $z^4 - 12z^3 + 62z^2 - 172z + 221 = 0$.

   (b) Find all the roots of the equation $log z = \left(\frac{17\pi}{2}\right)i$.

   (c) Evaluate
      i. $(1-i)^{\pi}$,
      ii. $(1-i)^{1+i}$.

The End.