

The Open University of Sri Lanka
 B.Sc/B.Ed. Degree Programme
 Final Examination - 2017/2018
 Pure Mathematics - Level 04
 PUU2140/PUE4140- Sequence and Series



Duration: - Two Hours

Date: - 17-09-2018

Time: -1.30 p.m -3.30 p.m.

Answer Four questions only.

- 1 (a) Prove that the sequence $\left\langle \frac{10n-1}{50-n^2} \right\rangle$ is bounded. Write down the greatest lower bound and the least upper bound. Find whether the sequence is monotonic. Justify your answer.
- (b) Let $a_n = n - 1 - \sqrt{n^2 - 1}$. Using the definition of convergent sequence, prove that the sequence $\langle a_n \rangle$ is convergent.
- (c) Suppose the sequence $\langle b_n \rangle$ is convergent and the set $\{b_n : n \in \mathbb{N}\}$ is finite. Prove that there exists $n_0 \in \mathbb{N}$ such that $b_n = c$ for all $n \geq n_0$, where c is a real number.
2. Let a and D be positive real numbers. The sequence $\langle s_n \rangle$ is defined recursively by,
 $s_1 = a$, $s_{n+1} = \frac{1}{2} \left(s_n + \frac{D}{s_n} \right)$ for $n \in \mathbb{N}$.
- (i) Prove that $s_{n+1} \geq \sqrt{D}$ for $n \geq 2$.
- (ii) Prove that $s_{n+1} \leq s_n$ for $n \geq 2$.
- (iii) Deduce that $\langle s_n \rangle$ is convergent and find its limit.
- (iv) Prove that $\langle s_n \rangle$ is a constant sequence iff $s_1 = \sqrt{D}$.
3. (i) Show that the sequence $\left\langle \sum_{k=1}^n \frac{1}{(k-1)!} \right\rangle$ is bounded above. (Recall that $0! = 1$)
- (ii) Show that the sequence $\left\langle \left(1 + \frac{1}{n}\right)^n \right\rangle$ is strictly increasing.
 (Hint: consider $c_k = 1 + \frac{1}{n}$ for $k = 1, 2, 3, \dots, n$ and $c_{n+1} = 1$ and use the arithmetic geometric inequality $\frac{1}{n+1} (c_1 + c_2 + \dots + c_{n+1}) > (c_1 c_2 \dots c_{n+1})^{\frac{1}{n+1}}$ for positive reals c_1, c_2, \dots, c_{n+1} .)
- (iii) Show that the sequence $\langle \left(1 + \frac{1}{n}\right)^n \rangle$ is bounded above.

4 (a) Prove that the geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ converge iff $|r| < 1$.

(b) Let r be a real number such that $0 < r < 1$ and $m \in \mathbb{N}$.

Prove that $0 < r^m < \frac{r}{m(1-r)}$.

(c) Prove that $\sum_{i=(n+1)}^{2n} \frac{1}{i} \geq \frac{1}{2}$, for any $n \in \mathbb{N}$.

Deduce that $\sum_{i=1}^{2^m} \frac{1}{i} \geq 1 + \frac{m}{2}$ for any $m \in \mathbb{N}$.

5 (a) Find whether each of the following series is convergent or divergent. Justify your answer.

(i) $\sum_{n=1}^{\infty} \sqrt{n+1} - \sqrt{n+2}$

(ii) $\sum_{n=1}^{\infty} \frac{(n^2+10)}{(1-2n+n^2)}$

(iii) $\sum_{n=1}^{\infty} \frac{1}{(2^n+(-2)^n+2)}$

(iv) $\sum_{n=1}^{\infty} \left(\frac{2^n \cdot n^{100}}{3^n} \right)$

(v) $\sum_{n=1}^{\infty} a_n$, where $a_n = \begin{cases} r^{n+1}, & n \text{ is even} \\ r^{n-1}, & n \text{ is odd} \end{cases}$ and $0 < r < 1$.

6 (a) Find the lim sup and lim inf of each of the following sequences.

(i) $\left\langle \sin\left(\frac{n\pi}{3}\right) \right\rangle$ (ii) $\left\langle (-1)^n \left(2 + \frac{3}{n}\right) \right\rangle$ (iii) $\left\langle \sum_{k=1}^n \frac{(-1)^k}{2^k} \right\rangle$

(b) Find the radius of convergence of each of the following power series.

(i) $\sum_{n=1}^{\infty} \left(\frac{n+1}{n+2}\right) x^n$ (ii) $\sum_{n=1}^{\infty} a_n x^n$, where $a_n = \begin{cases} \frac{1}{2^n}, & n \text{ is even} \\ \frac{1}{n}, & n \text{ is odd} \end{cases}$