The Open University of Sri Lanka
B.Sc/B.Ed. Degree Programme
Final Examination - 2017/2018
Pure Mathematics - Level 04
PEU4315/PUU2141/PUE4141 Continuous Functions

Duration: Two Hours

Date: 30.09.2018 Time: 01.30pm - 03.30pm

Answer <u>FOUR</u> Questions <u>ONLY</u>.

1. (a) Let $E = \left\{ \frac{5n}{3n+1} : n \in \mathbb{N} \right\}$. Prove that $\frac{5}{3}$ is a limit point of E.

   (b) Let $f : (0, \infty) \to (0, \infty)$ be given by $f(x) = \sqrt{x} + 2$, for each $x \in (0, \infty)$. Prove that for each $a \in (0, \infty)$, $\lim_{x \to a} f(x) = \sqrt{a} + 2$.

   (c) Let $f(x) = x^2 + 4x + 3$ for each $x \in \mathbb{R}$. Prove that $\lim_{x \to 1} f(x) \neq 6$ using the definition.

2. (a) Let $f : [1, \infty) \to \mathbb{R}$ be a function such that $\lim_{x \to 2} f(x)$ exists and $f(x) \leq 2018$ for each $x \in [1, \infty)$. Prove that $\lim_{x \to 2} f(x) \leq 2018$.

   (b) Suppose $\lim_{x \to c} f(x) = 0$ and there exist $k > 0$ and $\delta_1 > 0$ such that $|g(x)| \leq k$ for each x such that $0 < |x - c| < \delta_1$. Prove that $\lim_{x \to c} f(x)g(x) = 0$.

   (c) Find two functions f and g defined on $(0, 1)$ such that $\lim_{x \to 1^-} g(x)$ does not exist and $\lim_{x \to 1^-} f(x)g(x) = 0$. Justify your answer.

3. (a) Let $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & \text{if } x > 0; \\ \frac{1}{x}, & \text{if } x < 0. \end{cases}$
   Use Sandwich Theorem to show that $\lim_{x \to 0^+} f(x) = 0$.

   (b) Let $f(x) = \frac{5x^2+2}{7x^2+6x+1}$ for each $x \in (0, \infty)$. Prove that $\lim_{x \to \infty} f(x) = \frac{5}{7}$.

   (c) Let $f(x) = \frac{2x^4+1}{3x^2+1}$, $x \in \mathbb{R}$. Prove that $\lim_{x \to \infty} f(x) = \infty$.

4. (a) Let $f(x) = \begin{cases} 2x+3, & \text{if } x \in \mathbb{Q}; \\ -2x+3, & \text{if } x \in \mathbb{Q}^c. \end{cases}$

   Is f continuous at 0? Prove your answer.

   (b) Let $f(x) = \frac{3x+1}{x^2+5}$, $x \in \mathbb{R}$. Prove that f is continuous at 1.

   (c) Find two functions f and g such that both f, g are continuous at 0 and only at 0 but there exist $\alpha, \beta \in \mathbb{R}\backslash\{0\}$ such that $\alpha f + \beta g$ is continuous at 2018.

5. (a) Let $f(x) = \begin{cases} x^3+2x^2+3x+1, & \text{if } x \geq 1; \\ 21, & \text{if } x < 1. \end{cases}$

   Prove that f is right-continuous at 1. Is f continuous at 1? Justify your answer.

   (b) Let $h(x) = \begin{cases} 2x^2+5, & \text{if } x \leq 0; \\ x, & \text{if } x > 0. \end{cases}$

   Prove that h is left-continuous at 0 and is not continuous at 0.

   (c) Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 25, & \text{if } x > 1; \\ 12, & \text{if } x \leq 1; \end{cases}$ and $g(x) = \begin{cases} 2018, & \text{if } x = 25; \\ 15, & \text{if } x > 25; \\ 10, & \text{if } x < 25. \end{cases}$

   Find $\lim_{x\to 1^+} f(x)$, $\lim_{x\to 25^+} g(x)$ and $\lim_{x\to 1^+} g(f(x))$ with justification.

6. (a) Show that $f(x) = \sin x$ is uniformly continuous on $[0, \infty)$. (Use the fact that $|\sin x| \leq |x|$ for each $x \in \mathbb{R}$.)

   (b) Is the function $f(x) = \frac{x-1}{x+1}$ uniformly continuous on $[0, 2]$? Justify your answer.

   (c) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2 + 1$ for each $x \in \mathbb{R}$. Show that f is uniformly continuous on $(0, 1]$. Does this imply f is uniformly continuous on $\mathbb{R}$? Justify your answer.

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